

# The Structure of Chromatic Polynomials of Planar Triangulation Graphs and Implications for Chromatic Zeros and Asymptotic Limiting Quantities

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We present an analysis of the structure and properties of chromatic polynomials  $P(G_{pt,\vec{m}}, q)$  of one-parameter and multi-parameter families of planar triangulation graphs  $G_{pt,\vec{m}}$ , where  $\vec{m} = (m_1, \dots, m_p)$  is a vector of integer parameters. We use these to study the ratio of  $|P(G_{pt,\vec{m}}, \tau + 1)|$  to the Tutte upper bound  $(\tau - 1)^{n-5}$ , where  $\tau = (1 + \sqrt{5})/2$  and  $n$  is the number of vertices in  $G_{pt,\vec{m}}$ . In particular, we calculate limiting values of this ratio as  $n \rightarrow \infty$  for various families of planar triangulations. We also use our calculations to study zeros of these chromatic polynomials. We study a large class of families  $G_{pt,\vec{m}}$  with  $p = 1$  and  $p = 2$  and show that these have a structure of the form  $P(G_{pt,m}, q) = c_{G_{pt},1}\lambda_1^m + c_{G_{pt},2}\lambda_2^m + c_{G_{pt},3}\lambda_3^m$  for  $p = 1$ , where  $\lambda_1 = q - 2$ ,  $\lambda_2 = q - 3$ , and  $\lambda_3 = -1$ , and  $P(G_{pt,\vec{m}}, q) = \sum_{i_1=1}^3 \sum_{i_2=1}^3 c_{G_{pt},i_1 i_2} \lambda_{i_1}^{m_1} \lambda_{i_2}^{m_2}$  for  $p = 2$ . We derive properties of the coefficients  $c_{G_{pt},\vec{i}}$  and show that  $P(G_{pt,\vec{m}}, q)$  has a real chromatic zero that approaches  $(1/2)(3 + \sqrt{5})$  as one or more of the  $m_i \rightarrow \infty$ . The generalization to  $p \geq 3$  is given. Further, we present a one-parameter family of planar triangulations with real zeros that approach 3 from below as  $m \rightarrow \infty$ . Implications for the ground-state entropy of the Potts antiferromagnet are discussed.

## I. INTRODUCTION

In this paper we present exact results on the structure and properties of chromatic polynomials  $P(G_{pt,\vec{m}}, q)$  of one-parameter and multi-parameter families of planar triangulation  $(pt)$  graphs  $G_{pt,\vec{m}}$ , where  $\vec{m} = (m_1, \dots, m_p)$  is a vector of integer parameters. Our results substantially generalize our previous study in [1]. In standard notation we let  $G = (V, E)$  be a graph with vertex and edge sets  $V$  and  $E$ , and denote the number of vertices and edges as  $n = n(G) = |V|$  and  $e(G) = |E|$ , respectively. The resultant set of faces is denoted  $F(G)$ , with cardinality  $f(G) = |F(G)|$ . We recall the definitions of a planar graph  $G$  as one that can be drawn in a plane without any crossing edges, and a triangulation as a graph all of whose faces are triangles. For an arbitrary graph  $G$ , the chromatic polynomial  $P(G, q)$  enumerates the number of ways of associating  $q$  colors with the vertices of  $G$ , subject to the constraint that adjacent vertices have different colors (called a proper  $q$ -coloring of  $G$ ) [2]. The minimum number of colors for a proper  $q$ -coloring of a graph  $G$  is the chromatic number of  $G$ , denoted  $\chi(G)$ . Without loss of generality, we restrict here to graphs  $G$  that are connected, have no multiple edges, and have no loops (where a loop is defined as an edge that connects a vertex back to itself). Multiple edges are also excluded by our restriction here to triangulation graphs, since they produce faces that are not triangles. An important identity is  $P(G, q) = Z_{PAF}(G, q, 0)$ , where  $Z_{PAF}(G, q, T)$  denotes the partition function of the Potts antiferromagnet (PAF) on the graph  $G$  at temperature  $T$ . Because of this identity, properties of chromatic polynomials are of interest both for mathematical graph theory and for statistical physics. A particularly significant feature of the Potts antiferromagnet is the fact that it generically exhibits nonzero ground-state (i.e., zero-temperature) entropy per vertex for sufficiently large  $q$  on a given graph.

The chromatic polynomial of a graph  $G$  may be computed via the deletion-contraction relation  $P(G, q) = P(G - e, q) - P(G/e, q)$ , where  $G - e$  denotes  $G$  with an edge  $e$  deleted and  $G/e$  denotes the graph obtained by deleting  $e$  and identifying the vertices that it connected. From this follows the cluster formula  $P(G, q) = \sum_{G' \subseteq G} (-1)^{e(G')} q^{k(G')}$ , where  $G' = (V, E')$  with  $E' \subseteq E$  and  $k(G')$  denotes the number of connected components of  $G'$ . Via this relation, one can generalize  $q$  from positive integers to real and complex numbers, as is necessary when analyzing zeros of  $P(G, q)$ , called chromatic zeros. For a general graph  $G$  or family of graphs  $G_{\vec{m}}$ , the calculation of the chromatic polynomial takes an exponentially long time. Planar triangulations comprise a class that is particularly amenable to analysis, as we shall show. Since a triangulation graph  $G_t$  (whether planar or not) contains at least one triangle,  $P(G_t, q)$  always contains the factor  $P(K_3, q) = q(q-1)(q-2)$ . (Here,  $K_n$  is the complete graph with  $n$  vertices, defined as the graph such that each vertex is adjacent to every other vertex by

an edge.)

An interesting upper bound on an evaluation of a chromatic polynomial of a planar triangulation was derived by Tutte [3] (see also [4],[5]), namely

$$0 < |P(G_{pt}, \tau + 1)| \leq U(n(G_{pt})) , \quad (1.1)$$

where  $\tau = (1 + \sqrt{5})/2$  is the golden ratio and

$$U(n) = \tau^{5-n} = (\tau - 1)^{n-5} . \quad (1.2)$$

As in [1], it is natural to define the ratio

$$r(G_{pt}) \equiv \frac{|P(G_{pt}, \tau + 1)|}{U(n(G_{pt}))} , \quad (1.3)$$

which is bounded above by 1. In this paper we shall present calculations of this ratio, and its limit as  $n \rightarrow \infty$ , for a number of different families of planar triangulations. In [1] we showed that if  $p = 1$ , then, with  $\vec{m} \equiv m$ , if  $P(G_{pt,m}, q)$  involves only a single power of a polynomial,  $(\lambda_{G_{pt}})^m$  as in (2.8) below, it follows that  $r(G_{pt,m})$  approaches zero exponentially rapidly as  $m \rightarrow \infty$ . In [1] we also constructed one-parameter families  $G_{pt,m}$  for which  $P(G_{pt,m}, q)$  is a sum of powers of certain terms  $\lambda_i$  with  $i = 1, 2, 3$ , given below in (3.1), then  $r(G_{pt,m})$  may approach a finite nonzero constant as  $m \rightarrow \infty$ . We generalize these results here to the families  $G_{pt,\vec{m}}$  with  $p \geq 2$ . We also exhibit a  $p = 1$  family, denoted  $F_m$ , for which  $P(F_m, q)$  is a sum of nonpolynomial  $\lambda_{F,i}$ s,  $i = 1, 2, 3$ , and show that for this family,  $r(F_m)$  decreases to zero (exponentially rapidly) as  $m \rightarrow \infty$ . It should be noted that the Tutte upper bound is satisfied as an equality by the triangle,  $K_3$ , but for planar triangulations with higher  $n$ , the upper bound is realized as a strict inequality.

Part of our work concerns zeros of chromatic polynomials  $P(G_{pt,\vec{m}}, q)$  for families of planar triangulations  $G_{pt,\vec{m}}$ , i.e., chromatic zeros of these graphs. An interesting aspect of this study relates to an empirical observation made by Tutte in connection with his upper bound, namely that for a planar triangulation  $G_{pt}$ ,  $P(G_{pt}, q)$  typically has a real zero close to  $q = \tau + 1 = 2.6180339887\dots$ . This observation has been somewhat mysterious over the years, for at least two reasons. First, Tutte's upper bound (1.1) does not imply that a  $P(G_{pt}, q)$  need have a zero near to  $\tau + 1$ . Second, it is known that for an arbitrary (loopless) graph  $G$ ,  $P(G, q)$  cannot vanish exactly at  $q = \tau + 1 = (3 + \sqrt{5})/2$ . We recall the proof. If  $P(G, (3 + \sqrt{5})/2)$  were zero, then, since  $P(G, q)$  is a polynomial in  $q$ , it would have the factor  $[q - (3 + \sqrt{5})/2]$ . But since  $P(G, q)$  has rational coefficients (actually, integer coefficients, but all we use here is the property that they are rational), it would therefore also have to contain a factor involving the algebraic conjugate root, namely  $[q - (3 - \sqrt{5})/2]$ . However, this would imply

that  $P(G, q)$  would also vanish at  $q = (3 - \sqrt{5})/2 = 0.381966\dots$ , but this is impossible, since this point lies in an interval  $(0,1)$  where  $P(G, q)$  cannot vanish [6, 7]. As part of our analysis here, we shed some light on this mystery by constructing families of planar triangulation graphs  $G_{pt, \vec{m}}$  with  $p = 1$  and  $p = 2$  for which the chromatic polynomials have the respective forms  $P(G_{pt, m}, q) = c_{G_{pt}, 1} \lambda_1^m + c_{G_{pt}, 2} \lambda_2^m + c_{G_{pt}, 3} \lambda_3^m$  for  $p = 1$  (with  $m_1 \equiv m$ ), where  $\lambda_1 = q - 2$ ,  $\lambda_2 = q - 3$ , and  $\lambda_3 = -1$ , and  $P(G_{pt, \vec{m}}, q) = \sum_{i_1=1}^3 \sum_{i_2=1}^3 c_{G_{pt}, i_1 i_2} \lambda_{i_1}^{m_1} \lambda_{i_2}^{m_2}$  for  $p = 2$ . We derive properties of the coefficients  $c_{G_{pt}, \vec{i}}$  and show that  $P(G_{pt, \vec{m}}, q)$  has a real chromatic zero that approaches  $(1/2)(3 + \sqrt{5})$  as one or more of the  $m_i \rightarrow \infty$ . We give the generalization of this result to  $p \geq 3$ . Further, we construct a one-parameter family of planar triangulations of this type with real zeros that approach 3 from below as  $m \rightarrow \infty$ .

## II. GENERAL PROPERTIES OF ONE-PARAMETER FAMILIES OF PLANAR TRIANGULATIONS

### A. General

We have constructed and studied various one-parameter families of planar triangulations  $G_{pt, m}$  that can be built up in an inductive (recursive) manner. (Here and below, for families where  $\vec{m}$  is one-dimensional, we set  $m_1 \equiv m$  to simplify the notation.) In this section we derive general properties of the chromatic polynomials of these families of planar triangulations. For our families, the number of vertices is linearly related to  $m$ ,

$$n(G_{pt, m}) = \alpha m + \beta, \quad (2.1)$$

where  $\alpha$  and  $\beta$  are constants that depend on the type of family. We recall the Euler relation  $|V(G)| - |E(G)| + |F(G)| = \chi_E = 2$  for a graph  $G$  embedded on a surface of genus 0, such as the plane, where  $\chi_E$  is the Euler characteristic. In general, for a planar graph each of whose faces has  $p$  sides,  $n(G)$ ,  $e(G)$ , and  $f(G)$  satisfy the relations  $e(G) = p(n(G) - 2)/(p - 2)$  and  $f(G) = 2(n(G) - 2)/(p - 2)$ . For the case of interest here, namely planar triangulation graphs, where each face is a triangle, it follows that

$$e(G_{pt}) = 3(n(G_{pt}) - 2) \quad (2.2)$$

and

$$f(G_{pt}) = 2(n(G_{pt}) - 2), \quad (2.3)$$

so that  $e(G_{pt}) = (3/2)f(G_{pt})$ .

In our present study we will make use of several results that we derived in [1]. First, since  $U(n) \rightarrow 0$  as  $n \rightarrow \infty$  and since  $m$  is proportional to  $n$ , it follows that for these families of planar triangulations,

$$\lim_{m \rightarrow \infty} P(G_{pt,m}, \tau + 1) = 0 . \quad (2.4)$$

Second, given the upper bound (1.1) and the fact that  $U(n)$  approaches zero exponentially fast as  $n \rightarrow \infty$ , it follows that

$$P(G_{pt,m}, \tau + 1) \text{ approaches zero exponentially fast as } m \rightarrow \infty. \quad (2.5)$$

We recall two definitions that apply to any graph: (i) the degree  $d(v_i)$  of a vertex  $v_i \in V$  is the number of edges that connect to it, and (ii) a  $k$ -regular graph is a graph for which all vertices have degree  $k$ . Since a triangulation graph  $G_t$  is not, in general,  $k$ -regular, it is useful to define an effective vertex degree in the limit  $|V| \rightarrow \infty$ . For this purpose, we introduce, as in our earlier work, the notation  $\{G\}$  for the formal limit  $n \rightarrow \infty$  of a family of graphs  $G$ . We define

$$\begin{aligned} d_{eff}(\{G\}) &= \lim_{|V| \rightarrow \infty} \frac{2|E|}{|V|} \\ &= \frac{\sum_i n_i d_i}{|V|} , \end{aligned} \quad (2.6)$$

where for a given  $G$ ,  $n_i$  denotes the number of vertices with degree  $d_i$  and  $n(G) \equiv |V|$ . Substituting (2.2) in (2.6), we obtain

$$d_{eff}(\{G_{pt}\}) = 6 . \quad (2.7)$$

We will use this below, in Sect. XIX.

## B. Families with $P(G_{pt,m}, q)$ Consisting of a Power of a Single Polynomial

There are several ways of constructing one-parameter families of planar triangulations. One method that we have used is the following, which produces families for which the chromatic polynomial involves a single power of a polynomial in  $q$ . Start with a basic graph  $G_{pt,1}$ , drawn in the usual explicitly planar manner. The outer edges of this graph clearly form a triangle,  $K_3$ . Next pick an interior triangle in  $G_{pt,1}$  and place a copy of  $G_{pt,1}$  in this triangle so that the intersection of the resultant graph with the original  $G_{pt,1}$  is the triangle chosen. Denote this as  $G_{pt,2}$ . Continuing in this manner, one constructs  $G_{pt,m}$  with  $m \geq 3$ . The chromatic polynomial  $P(G_{pt,2}, q)$  is calculated from  $P(G_{pt,1}, q)$  by using the  $s = 3$  special

case of the complete-graph intersection theorem. This theorem states that if for two graphs  $G$  and  $H$  (which are not necessarily planar or triangulations), the intersection  $G \cap H = K_s$  for some  $s$ , then  $P(G \cup H, q) = P(G, q)P(H, q)/P(K_s, q)$ . (Note that  $P(K_s, q) = \prod_{j=0}^{s-1} (q - j)$ .)

It follows that for planar triangulations formed in this iterative manner, the chromatic polynomial has the form

$$P(G_{pt,m}, q) = c_{G_{pt}} (\lambda_{G_{pt}})^m. \quad (2.8)$$

where the coefficient  $c_{G_{pt}}$  and the term  $\lambda_{G_{pt}}$  are polynomials in  $q$  that do not depend on  $m$ . Here and below, it is implicitly understood that  $m \geq m_{min}$ , where  $m_{min}$  is the minimal value of  $m$  for which the family  $G_{pt,m}$  is well defined. Since  $G_{pt}$  contains at least one triangle,  $K_3$ , the coefficient  $c_{G_{pt}}$  contains (and may be equal to)  $P(K_3, q) = q(q-1)(q-2)$ . The chromatic number of  $G_{pt}$  may be 3 or 4. In the case of a planar triangulation which is a strip of the triangular lattice of length  $m$  vertices with cylindrical boundary conditions, to be discussed below, an alternate and equivalent way to construct the  $(m+1)$ 'th member of the family is simply to add a layer of vertices to the strip at one end.

### C. Families with $P(G_{pt,m}, q)$ Consisting of Powers of Several Functions

We have also devised methods to obtain families of planar triangulations  $G_{pt,\vec{m}}$  with the property that the chromatic polynomial is a sum of more than one power of a function of  $q$ . We begin with the simplest case,  $p = 1$ , i.e., one-parameter families and then discuss families with  $p \geq 2$ . For one-parameter families, we find the general structure

$$P(G_{pt,m}, q) = \sum_{j=1}^{j_{max}} c_{G_{pt},j} (\lambda_{G_{pt},j})^m, \quad (2.9)$$

where  $m \geq m_{min}$  and the  $c_{G_{pt},j}$  and  $\lambda_{G_{pt},j}$  are certain coefficients and functions depending on  $q$  but not on  $m$ . Here we use the label  $G_{pt}$  to refer to the general family of planar triangulations  $G_{pt,m}$ . We will describe these methods below. Parenthetically, we recall that the form (2.9) is a general one for one-parameter recursive families of graphs, whether or not they are planar triangulations [11], [8]. For (2.9) evaluated at a given value  $q = q_0$ , as  $m \rightarrow \infty$ , and hence  $n \rightarrow \infty$ , the behavior of  $P(G_{pt,m}, q)$  is controlled by which  $\lambda_{G_{pt},j}$  is dominant at  $q = q_0$ , i.e., which of these has the largest magnitude  $|\lambda_{G_{pt},j}(q_0)|$ . For our purposes, a  $q_0$  of major interest is  $\tau + 1$ , since the Tutte upper bound (1.1) applies for this value. We denote the  $\lambda_{G_{pt},j}$  that is dominant at  $q = \tau + 1$  as  $\lambda_{G_{pt},dom}$ . Clearly, if  $P(G_{pt,m}, q)$  involves only a single power, as in (2.8), then  $\lambda_{G_{pt},dom} = \lambda_{G_{pt}}$ .

As in earlier works [8–10], it can be convenient to obtain the chromatic polynomials  $P(G_{pt,m}, q)$  via a Taylor series expansion, in an auxiliary variable  $x$ , of a generating function

$\Gamma(G_{pt}, q, x)$ . Below, we will have occasion to use this method for the family  $F_m$  (see (18.4)). Both the form (2.9) and the expression via a generating function are equivalent to the property that  $P(G_{pt,m}, q)$  satisfies a recursion relation, for  $m \geq j_{max} + m_{min}$ :

$$P(G_{pt,m}, q) + \sum_{j=1}^{j_{max}} b_{G_{pt},j} P(G_{pt,m-j}, q) = 0 , \quad (2.10)$$

where the  $b_{G_{pt},j}$ ,  $j = 1, \dots, j_{max}$ , are given by

$$1 + \sum_{j=1}^{j_{max}} b_j x^j = \prod_{j=1}^{j_{max}} (1 - \lambda_{G_{pt},j} x) . \quad (2.11)$$

Thus,

$$b_{G_{pt},1} = - \sum_{j=1}^{j_{max}} \lambda_{G_{pt},j} , \quad (2.12)$$

$$b_{G_{pt},2} = \sum_{j=1, k=1, j \neq k}^{j_{max}} \lambda_{G_{pt},j} \lambda_{G_{pt},k} , \quad (2.13)$$

and so forth, up to

$$b_{G_{pt},j_{max}} = (-1)^{j_{max}} \prod_{j=1}^{j_{max}} \lambda_{G_{pt},j} . \quad (2.14)$$

#### D. Asymptotic Behavior as $m \rightarrow \infty$

In [1] we discussed the asymptotic behavior of the chromatic polynomials as  $m \rightarrow \infty$ . In both the cases of Eq. (2.8) and (2.9), a single power  $[\lambda_{G_{pt},j}]^m$  dominates the sum as  $m \rightarrow \infty$ . For a member of a one-parameter family of planar triangulations,  $G_{pt,m}$ , we use the notation  $r(G_{pt,m})$  for the ratio (1.3), and we define

$$r(G_{pt,\infty}) \equiv \lim_{m \rightarrow \infty} r(G_{pt,m}) . \quad (2.15)$$

We define the (real, non-negative) constant  $a_{G_{pt}}$  as [1]

$$a_{G_{pt}} = \lim_{n \rightarrow \infty} [r(G_{pt,m})]^{1/n} = \frac{|\lambda_{G_{pt},dom}(\tau + 1)|^{1/\alpha}}{\tau - 1} . \quad (2.16)$$

We showed in [1] that if  $j_{max} = 1$ , then  $a_{G_{pt}} < 1$  and hence for the classes of  $G_{pt,m}$  under consideration, (i)  $r(G_{pt,\infty}) = 0$  and (ii)  $r(G_{pt,m})$  decreases toward zero exponentially rapidly

as a function of  $m$  and  $n$  as  $m \rightarrow \infty$ . Note that this does not imply that  $P(G_{pt,m}, q)$  has a zero that approaches  $q = \tau + 1$  as  $m, n \rightarrow \infty$ .

For one-parameter families of planar triangulation graphs  $G_{pt,m}$  where  $P(G_{pt,m}, q)$  has the form (2.9) with  $j_{max} \geq 2$ , a consequence of the Tutte upper bound (1.1) is that as  $m \rightarrow \infty$ , any contribution  $c_{G_{pt},j}(\lambda_{G_{pt},j})^m$  in (2.9), when evaluated at  $q = \tau + 1$ , must be less than or equal in magnitude to  $(\tau - 1)^{n-5}$ . Therefore, for a given  $j$  in this case, either the coefficient  $c_{G_{pt},j}$  vanishes for  $q = \tau + 1$  or, if this coefficient does not vanish at  $q = \tau + 1$ , then, taking into account the relation (2.1), it follows that

$$\frac{|\lambda_{G_{pt},j}|^{1/\alpha}}{\tau - 1} \leq 1 \quad \text{at } q = \tau + 1 \quad \forall j. \quad (2.17)$$

If this inequality is realized as an equality, then  $r(G_{pt,\infty})$  is a nonzero constant, which necessarily lies in the interval  $(0, 1)$ , so that  $a_{G_{pt}} = 1$ . For the families  $G_{pt,m}$  for which  $m$  and  $n$  are linearly related, as specified in (2.1), this type of behavior occurs if and only if, when  $P(G_{pt,m}, q)$  is evaluated at  $q = \tau + 1$ , the (necessarily) dominant  $\lambda_{G_{pt},j}$  (with nonvanishing coefficient  $c_{G_{pt},j}$ ), is equal to  $\tau - 1$  in magnitude, i.e.,  $|\lambda_{G_{pt},dom}| = \tau - 1$  at  $q = \tau + 1$ . This is true, in particular, if this  $\lambda_{G_{pt},j} = q - 2$ . In the structural form (3.16) below, we shall label this as the  $j = 1$  term.

It is a general property that if a graph  $G$  contains a complete graph  $K_p$  as a subgraph, then  $P(G, q)$  contains the factor  $P(K_p, q)$ . In particular, a triangulation graph, whether planar or not, has the factor  $P(K_3, q)$  and a planar triangulation may also contain a  $K_4$ . (However, by Kuratowski's Theorem, a planar graph may not contain a  $K_p$  with  $p \geq 5$ .) Thus, for a planar triangulation  $G_{pt}$ ,  $P(G_{pt}, q) = 0$  for  $q = 0, 1, 2$ . If  $G_{pt} \supseteq K_4$ , then  $P(G_{pt}, q)$  also vanishes at  $q = 3$ . If  $P(G_{pt}, q)$  has the form of a single power, given as Eq. (2.8), then these factors are explicit. If, however,  $P(G_{pt}, q)$  has the form of a sum of  $j_{max} \geq 2$  powers  $[\lambda_{G_{pt},j}]^m$ , then the conditions that  $P(G_{pt}, q)$  vanish at  $q = 0, 1, 2$  imply relations between the various terms. Moreover, the condition that  $P(G_{pt,m}, \tau + 1)$  obeys the Tutte upper bound (1.1) also implies conditions on the structure of this chromatic polynomial. We derive these next.



### III. PROPERTIES OF A CLASS OF $G_{pt,m}$ WITH $P(G_{pt,m}, q)$ HAVING $j_{max} = 3$ AND CERTAIN $\lambda_{G_{pt},j}$

#### A. Structure of Coefficients $c_{G_{pt},j}$ in $P(G_{pt,m}, q)$

For a large class of one-parameter families of planar triangulations that we have constructed and studied, for which  $P(G_{pt,m}, q)$  has the form (2.9), we find that (i)  $j_{max} = 3$  and (ii) the  $\lambda_{G_{pt},j} \equiv \lambda_j$  with  $j = 1, 2, 3$  have the form

$$\lambda_1 = q - 2, \quad \lambda_2 = q - 3, \quad \lambda_3 = -1 . \quad (3.1)$$

For this class of planar triangulations, we can derive some general results concerning the functional form of the coefficients  $c_{G_{pt},j}$  (where we will often suppress the subscript  $pt$  on  $G_{pt}$  where the meaning is obvious). Using the general form (2.9) with  $j_{max} = 3$  and these  $\lambda_j$ 's, we can derive the following identities. The fact that  $P(G_{pt}, 0) = 0$  implies that

$$c_{G,1}(-2)^m + c_{G,2}(-3)^m + c_{G,3}(-1)^m = 0 . \quad (3.2)$$

where for ease of notation we suppress the subscript  $pt$  on  $G_{pt}$  here and in related equations. Since this equation must hold for arbitrary  $m$  (understood implicitly to be an integer in the range  $m \geq m_{min}$ , where  $m_{min}$  is the minimal value for which the family  $G_{pt,m}$  is well defined), it implies that  $c_{G,j} = 0$  for all  $j$ . Hence, for these families,

$$c_{G,j} \quad \text{contains the factor } q \text{ for } j = 1, 2, 3 . \quad (3.3)$$

The evaluation  $P(G_{pt}, 1) = 0$  reads

$$c_{G,1}(-1)^m + c_{G,2}(-2)^m + c_{G,3}(-1)^m = 0 \quad \text{at } q = 1 . \quad (3.4)$$

Since this equation must hold for arbitrary  $m \geq m_{min}$ , it implies two conditions on the evaluation of the coefficients at  $q = 1$ , namely

$$c_{G,2} = 0 \quad \text{and} \quad c_{G,1} + c_{G,3} = 0 \quad \text{at } q = 1 . \quad (3.5)$$

In particular, (3.5) implies that

$$c_{G,2} \quad \text{contains the factor } q - 1 . \quad (3.6)$$

The evaluation  $P(G_{pt}, 2) = 0$  reads

$$c_{G,1}0^m + [c_{G,2} + c_{G,3}](-1)^m = 0 \quad \text{at } q = 2 . \quad (3.7)$$

Since this equation holds for arbitrary  $m \geq m_{min}$ , it implies that

$$c_{G,2} + c_{G,3} = 0 \quad \text{at } q = 2 . \quad (3.8)$$

If a family  $G_{pt,m}$  which has  $m_{min} = 0$ , (3.7) and (3.8) together would also imply that  $c_{G,1} = 0$  at  $q = 2$ .

Continuing with  $P(G_{pt,m}, q)$  of the form (2.9) with (3.1), we next analyze the evaluation of  $P(G_{pt,m}, q)$  at  $q = \tau + 1$ , viz.,  $P(G_{pt,m}, \tau + 1)$ . Since  $\tau - 1 = 0.61803\dots$  and  $\tau - 2 = -0.381966$  are smaller than unity in magnitude, the first two terms in  $P(G_{pt,m}, \tau + 1)$  vanish exponentially rapidly as  $m$  increases. As regards the ratio  $r(G_{pt,m})$ , as  $m$  increases, the contribution of the first term to this upper bound approaches a constant, while the contribution of the second term vanishes exponentially rapidly. Given the relation (2.1), the Tutte upper bound also vanishes exponentially rapidly as a function of  $m$ . Therefore, in order for  $P(G_{pt,m}, \tau + 1)$  to satisfy the Tutte upper bound (1.1), it is necessary and sufficient that

$$c_{G,3} = 0 \quad \text{at } q = \tau + 1 . \quad (3.9)$$

This means that

$$c_{G,3} \quad \text{contains the factor} \quad q - \left( \frac{3 + \sqrt{5}}{2} \right) . \quad (3.10)$$

Given that a chromatic polynomial has rational (actually integer) coefficients as a polynomial in  $q$ , this means that  $c_{G,3}$  must also contain a factor involving the algebraically conjugate root, i.e.,

$$c_{G,3} \quad \text{contains the factor} \quad q - \left( \frac{3 - \sqrt{5}}{2} \right) . \quad (3.11)$$

Combining these, we derive the result that

$$c_{G,3} \quad \text{contains the factor} \quad q^2 - 3q + 1 . \quad (3.12)$$

Having proved these results, it is thus convenient to extract the factors explicitly and define

$$\kappa_{G,1} \equiv \frac{c_{G,1}}{q} , \quad (3.13)$$

$$\kappa_{G,2} \equiv \frac{c_{G,2}}{q(q-1)} , \quad (3.14)$$

and

$$\kappa_{G,3} \equiv \frac{c_{G,3}}{q(q^2 - 3q + 1)} . \quad (3.15)$$

For this class of planar triangulation graphs  $G_{pt,m}$ , we thus have the general structural formula

$$P(G_{pt,m}, q) = q \left[ \kappa_{G,1}(q-2)^m + \kappa_{G,2}(q-1)(q-3)^m \right]$$

$$+ \kappa_{G,3}(q^2 - 3q + 1)(-1)^m \Big] , \quad (3.16)$$

where  $m$  and  $n$  are related by (2.1). We observe that the form (3.16) satisfies the general results that we derived above for the evaluation at  $q = \tau + 1$ . Thus, if  $P(G_{pt,m}, q)$  has this form (3.16) with (2.1) and  $\alpha = 1$ , then

$$r(G_{pt,\infty}) = [q \kappa_{G,1}] \Big|_{q=\tau+1} \quad (3.17)$$

and hence

$$a(G_{pt}) = 1 . \quad (3.18)$$

The conditions on the coefficients  $c_{G,j}$ 's evaluated at  $q = 1$  and  $q = 2$  that we have derived, (3.5), together with the definitions (3.13)-(3.15), are equivalent to the following relations:

$$\kappa_{G,1} = \kappa_{G,3} \quad \text{at } q = 1 \quad (3.19)$$

and

$$\kappa_{G,2} = \kappa_{G,3} \quad \text{at } q = 2 . \quad (3.20)$$

For certain families of planar triangulations  $G_{pt,m} \equiv G_m$ , the chromatic number  $\chi(G_m)$  is 3 for even  $m$  and 4 for odd  $m$  or vice versa. In these cases, we can also derive another relation between the coefficients. Thus, if  $\chi(G_m) = 3$  for even  $m$  and  $\chi(G_m) = 4$  for odd  $m$ , then  $\kappa_{G,1} = \kappa_{G,3}$  at  $q = 3$ . On the other hand, if  $\chi(G_m) = 3$  for odd  $m$  and  $\chi(G_m) = 4$  for even  $m$ , then  $\kappa_{G,1} = -\kappa_{G,3}$  at  $q = 3$ . In the case of families  $G_m$  for which  $\chi(G_m) = 4$  for all  $m$ , we have

$$\kappa_{G,1} + \kappa_{G,3} (-1)^m = 0 \quad \text{at } q = 3 \quad \text{if } \chi(G_m) = 4 , \quad (3.21)$$

which implies

$$\kappa_{G,1} = \kappa_{G,3} = 0 \quad \text{at } q = 3 \quad \text{if } \chi(G_m) = 4 . \quad (3.22)$$

## B. Properties of Real Chromatic Zeros

Here we derive some properties of chromatic zeros of planar triangulation graphs  $G_{pt,m}$  for which the chromatic polynomial has the form (2.9). It is appropriate first to review some relevant properties of chromatic zeros of general graphs and planar triangulation graphs. For a general graph  $G$ , it is elementary that there are no negative chromatic zeros and that there are no chromatic zeros in the intervals  $(0,1)$  [6, 7]. The property that  $(0,1)$  is a zero-free interval for the chromatic polynomial implies that  $q = \tau + 1$  cannot be a chromatic zero for

any graph  $G$ , as noted above (independent of whether it is a planar triangulation or not). Another interval that has been proved to be free of chromatic zeros is  $(1, 32/27)$  [12, 13]. For an arbitrary graph  $G$ , let us denote the total number of subgraphs  $H \subseteq G$  that are triangles as  $N_t$ , and let  $n = n(G)$  and  $e = e(G)$ . It has been proved [14] that for an arbitrary graph  $G$  with  $n \geq 4$  vertices, if  $N_t < u(G)$ , where

$$u(G) = \frac{e(e - n) + n - 1}{2(n - 2)} , \quad (3.23)$$

then  $P(G, q)$  has complex zeros.

Specializing now to planar triangulation graphs, it has been proved that  $G_{pt}$  has no chromatic zeros in the interval  $(2, q_w)$  [15, 16], where  $q_w$  is the unique real zero of

$$\lambda_{TC} = q^3 - 9q^2 + 29q - 32 , \quad (3.24)$$

i.e.,

$$q_w = 3 - \frac{[12(9 + \sqrt{177})]^{1/3}}{6} + 4[12(9 + \sqrt{177})]^{-1/3} = 2.546602.. \quad (3.25)$$

We remark that  $q_w$  occurs as a chromatic zero of some planar triangulation graphs, in particular, the family comprised of cylindrical sections of the triangulation lattice with  $L_y = 3$ , or equivalently iterated octahedra. In 1992 Woodall conjectured that a planar triangulation has no chromatic zeros in the interval  $(q_m, 3)$ , where  $q_m = 2.6778146..$  is the unique real zero of  $q^3 - 9q^2 + 30q - 35$ , [15], but later he gave counterexamples to his conjecture involving one-parameter families of planar triangulations each of which has a real zero that approaches 3 from below as this parameter goes to infinity [17]. We also note that for a planar triangulation, substituting (2.2) into (3.23),

$$u(G) = \frac{(3n - 7)(2n - 5)}{2(n - 2)} \quad \text{for } G = G_{pt} . \quad (3.26)$$

Here we present some further results on chromatic zeros of planar triangulations. First, if  $P(G_{pt,m}, q)$  has the form (2.8) involving only a single power of a  $\lambda_{G_{pt}}$ , then its zeros are fixed, independent of  $m$ , and hence although it typically has a zero close to  $\tau + 1$ , this zero does not move as a function of  $m$ . However, if  $P(G_{pt,m}, q)$  has the multi-term form (3.16) with  $j_{max} = 3$  and the  $\lambda_j$ 's in (3.1), then it necessarily has a zero in the interval  $[q_w, 3)$  that approaches  $\tau + 1$  as  $m \rightarrow \infty$ . The proof of this is as follows. Let us assume that  $q$  is a real number in this interval  $[q_w, 3)$ . In the limit as  $m \rightarrow \infty$ , the first two terms, which are proportional to  $(q - 2)^m$  and  $(q - 3)^m$ , respectively, vanish (exponentially fast), so that

$$P(G_{pt,m}, q) \sim \epsilon_m + q(q^2 - 3q + 1)\kappa_{G_{pt},3}(-1)^m , \quad (3.27)$$

where  $\epsilon_m$  denotes the contribution of these first two terms. If  $\kappa_{G_{pt},3}$  happens to vanish at  $q = \tau + 1$ , then the result follows, since  $\epsilon_m \rightarrow 0$  as  $m \rightarrow \infty$ . If  $\kappa_{G_{pt},3} \neq 0$  at  $q = \tau + 1$ , then consider the limit as  $q \rightarrow \tau + 1$ , where we can write

$$\begin{aligned} \frac{(-1)^m P(G_{pt,m}, q)}{(\tau + 1) \kappa_{G_{pt},3}|_{q=\tau+1}} &= \frac{(-1)^m \epsilon_m}{(\tau + 1) \kappa_{G_{pt},3}|_{q=\tau+1}} + q^2 - 3q + 1 \\ &\equiv \delta_m + q^2 - 3q + 1 . \end{aligned} \quad (3.28)$$

To show that  $P(G_{pt}, q)$  has a zero that approaches  $q = \tau + 1$  as  $m \rightarrow \infty$ , we use the fact that  $P(G_{pt,m}, q)$  is a continuous function of  $q$  and solve (3.28) for  $q$ , subject to the condition that  $q \in [q_w, 3)$ , obtaining a consistent result with

$$q = \frac{1}{2} \left[ 3 + \sqrt{5 - 4\delta_m} \right] , \quad (3.29)$$

which approaches  $q = \tau + 1$  as  $m \rightarrow \infty$ . Note that the other zero at  $q = (1/2)(3 - \sqrt{5 - 4\delta_m})$ , is irrelevant because we assumed at the outset that  $q$  is in the interval  $[q_w, 3)$  and this other zero is outside this interval; in the vicinity of this other zero, the analysis does not apply because the terms proportional to  $(q - 2)^m$  and  $(q - 3)^m$  do not vanish as  $m \rightarrow \infty$ .

In 1.1 we have exhibited two one-parameter families of planar triangulation graphs with this property, namely  $B_m$  and  $H_m$ . We construct and analyze several more families of this type here. Thus, for these families, we have provided an understanding of why  $P(G_{pt,m}, q)$  has a chromatic zero near to  $\tau + 1$  and, furthermore, have proved that this zero approaches  $\tau + 1$  as  $m \rightarrow \infty$ . From the derivation above, it is evident that our result requires, for a given family  $G_{pt,m}$ , that  $m$  be sufficiently large. As is illustrated from numerical results presented below, for specific families of planar triangulations that we have studied,  $P(G_{pt,m}, q)$  has a real zero reasonably close to  $\tau + 1$  even for moderate values of  $m$ .

Our second result follows immediately from this analysis. With the same assumptions, we have observed that in the limit  $m \rightarrow \infty$ ,  $P(G_{pt,m}, q)$  has a real zero in the interval  $q \in [q_w, 3)$  if and only if  $c_{G_{pt},3}$  has a real zero in this interval,  $q \in [q_w, 3)$ . We know that there is at least one such zero, namely the one arising from the factor  $q^2 - 3q + 1$  in  $c_{G_{pt},3}$ . Therefore, in the limit  $m \rightarrow \infty$ ,  $P(G_{pt,m}, q)$  has another real zero in the interval  $q \in [q_w, 3)$  in addition to the one approaching  $\tau + 1$  if and only if  $\kappa_{G,3}$  has a real zero in this interval  $q \in [q_w, 3)$ . We will present several applications of these results below.

We remark that, with the same assumptions as above,

$$\lim_{m \rightarrow \infty} |P(G_{pt,m}, q)| = c_{G_{pt},3} . \quad (3.30)$$

Note that the limit  $\lim_{m \rightarrow \infty} P(G_{pt,m}, q)$  itself does not exist, because the term  $\lambda_3^m = (-1)^m$  factor has no limit as  $m \rightarrow \infty$ .

In [1], we investigated the question of whether for a planar triangulation graph  $G_{pt}$  it is true that the chromatic zero of  $G_{pt}$  nearest to  $\tau + 1$  is always real. We exhibited an example, with a graph we denoted  $G_{CM,1}$ , which is, to our knowledge, the first case for which the zero closest to  $\tau + 1$  is not real but instead the zeros closest to  $\tau + 1$  form a complex-conjugate pair. Our result is in agreement with a previous observation by Woodall that this graph has no real zero near to  $\tau + 1$  [18]. Since  $P(G_{CM,m}, q)$  has the form of (2.8) with a single  $\lambda$ , as  $m$  increases, its zeros are fixed and just increase in multiplicity, in contrast to the motion of chromatic zeros for families  $G_{pt,m}$  whose chromatic polynomials are of the form (2.9) with  $j_{max} > 1$ .

The value of  $q$  where the Tutte upper bound applies, namely  $q = \tau + 1 = (3 + \sqrt{5})/2$  is also a member of a sequence of numbers related to roots of unity, namely the Tutte-Beraha numbers,  $q_r$ . Thus, for a root of unity of the form  $z_r = e^{\pi i/r}$ , one defines  $q_r = (z_r + z_r^*)^2 = 4 \cos^2(\pi/r)$ . One has  $\tau + 1 = q_5$ . Parenthetically, we note that chromatic zeros have been studied for sections of triangular lattices with various boundary conditions that are not planar triangulations, either because they have at least one face that is not a triangle or because they are not planar (e.g., have toroidal or Klein-bottle boundary conditions). We refer the reader to [1] for references to some of these papers; here, in view of our focus on the Tutte upper bound (1.1) we restrict to planar triangulations.

### C. Properties of Complex Chromatic Zeros

As before, we consider a one-parameter of planar triangulation graphs  $G_{pt,m}$  such that  $P(G_{pt,m}, q)$  has the form (3.16). Here we give a general determination of the continuous accumulation set  $\mathcal{B}$  of chromatic zeros of  $P(G_{pt,m}, q)$  in the complex  $q$  plane in the limit  $m \rightarrow \infty$  (and, hence, owing to (2.1),  $n \rightarrow \infty$ ). If the zeros form a discrete set (some with multiplicities that go to infinity as  $n \rightarrow \infty$ ), then this locus is null. As in earlier work, we denote the formal limit of the family  $G_m$  as  $m \rightarrow \infty$  as  $\{G\}$ . In general, the locus  $\mathcal{B}$  may or may not intersect the real  $q$  axis. If it does, the maximal point where it intersects the real axis is denoted  $q_c(\{G\})$ . For a  $P(G, q)$  of the form (2.9), the curves comprising the locus  $\mathcal{B}$  are determined as the solutions of the equality in magnitude of the dominant  $\lambda_{G_{pt},j}$ , in accordance with general results for recursive functions [19]. These curves extend infinitely far from the origin if and only if such an equality can be satisfied as  $|q| \rightarrow \infty$ , as was discussed in [8],[20]-[22].

We now consider families of planar triangulations  $G_{pt,m}$  whose chromatic polynomials have the form (3.16). For these families, first, because  $\lambda_1 = q - 2$  is dominant for large  $q$  and is equal in magnitude to  $\lambda_3 = -1$  at  $q = 3$ , it follows that  $q_c(\{G_{pt}\}) = 3$ . Second,

as a consequence of the fact that the equality  $|q - 2| = |q - 3|$  holds for the infinite line  $\text{Re}(q) = 5/2$ , part of the boundary  $\mathcal{B}$  extends infinitely far away from the origin in the  $q$  plane, i.e., passes through the origin of the  $1/q$  plane. As is evident for specific families  $G_{pt,m}$ , as  $m$  and hence  $n$  go to infinity, the degree of one or more vertices also goes to infinity, so the fact that the magnitudes of zeros diverge is in accord with the upper bound  $|q| < b\Delta(G)$  obtained in [23] (with  $b \simeq 7.96$ ) and strengthened slightly in [24] (with  $b \simeq 6.91$ ), where  $\Delta(G)$  is the maximal degree of any vertex in  $G$ . Note, however, that the property that the degree of a vertex diverges as  $n \rightarrow \infty$  does not, by itself, imply that  $\mathcal{B}$  passes through the origin of the  $1/q$  plane. This is clear from the  $n \rightarrow \infty$  limit of wheel graphs, for which the central vertex  $v_{cent.}$  has degree  $d(v_{cent.}) \rightarrow \infty$ , but the locus  $\mathcal{B}$  has bounded support in the  $q$  plane [8, 25]. Continuing with our analysis of families of planar triangulations  $G_{pt,m}$  whose chromatic polynomials have the form (3.16), the  $\mathcal{B}$  separates the  $q$  plane into three regions, which we denote as  $R_j$ ,  $j = 1, 2, 3$ . These are defined as follows:

$$R_1 : \text{Re}(q) > \frac{5}{2} \quad \text{and} \quad |q - 2| > 1 , \quad (3.31)$$

$$R_2 : \text{Re}(q) < \frac{5}{2} \quad \text{and} \quad |q - 3| > 1 , \quad (3.32)$$

and

$$R_3 : |q - 2| < 1 \quad \text{and} \quad |q - 3| < 1 . \quad (3.33)$$

The boundaries between these regions are thus the two circular arcs

$$\mathcal{B}(R_1, R_3) : q = 2 + e^{i\theta} , -\frac{\pi}{3} < \theta < \frac{\pi}{3} \quad (3.34)$$

and

$$\mathcal{B}(R_2, R_3) : q = 3 + e^{i\phi} , \frac{2\pi}{3} < \phi < \frac{4\pi}{3} , \quad (3.35)$$

together with the semi-infinite vertical line segments

$$\mathcal{B}(R_1, R_2) = \{q\} : \quad \text{Re}(q) = \frac{5}{2} \quad \text{and} \quad |\text{Im}(q)| > \frac{\sqrt{3}}{2} . \quad (3.36)$$

These meet at the triple points

$$q_t, \quad q_t^* = \frac{5 \pm i\sqrt{3}}{2} . \quad (3.37)$$

A specific example of a locus  $\mathcal{B}$  that extends infinitely far from the origin in the  $q$  plane was studied in [8, 26] for the family  $B_m$ . Here we have generalized the result to all families of planar triangulations whose chromatic polynomials have the form (3.16).

We have constructed and studied a two-parameter family of planar triangulations  $D_{m_1, m_2}$ . By keeping one of the two indices  $m_1$  or  $m_2$  fixed, we have obtained a number of one-parameter families and have analyzed the chromatic polynomials for these. We have also

considered the special cases where one allows both  $m_1$  and  $m_2$  to vary, such that one is a linear function of the other. We have, in particular, analyzed the diagonal case where  $m_1 = m_2$ ,  $D_{m,m}$ , to be discussed below. For this family the chromatic polynomial  $P(D_{m,m}, q)$  has the form (2.9) with  $j_{max} = 6$  and a set of  $\lambda$ 's that are squares and cross products of those in (3.16). Since the equations defining the equality of magnitude of dominant  $\lambda$ 's can again be satisfied as  $1/q \rightarrow 0$ , it again follows, by the criteria of [8, 20, 37] that the locus  $\mathcal{B}$  extends infinitely far from the origin of the complex  $q$  plane.

In a different direction, we have also studied a family  $F_m$  (see below) for which the chromatic polynomial  $P(F_m, q)$  is of the form (3.16) with  $j_{max} = 3$ , but with terms  $\lambda_{F,j}$ ,  $j = 1, 2, 3$ , that are not simple polynomials, but instead are roots of a cubic equation, (18.16). In the  $m \rightarrow \infty$  limit, the continuous accumulation set  $\mathcal{B}$  for this family has bounded magnitude (does not extend infinitely far away from the origin in the  $q$  plane), as can be seen because the condition that defines  $\mathcal{B}$ , namely the equality in magnitude of two dominant  $\lambda_{F,j}$ 's, cannot be satisfied for arbitrarily great  $|q|$ .

#### IV. THE FAMILY $B_m$

In Ref. [1], we constructed and studied several one-parameter families of planar triangulations with chromatic polynomials of the multi-term form (3.16). (Some graphs in these families are listed in [27].) In view of the general factorizations for the coefficients  $c_{G_{pt,m},j}$  that we have proved here, it is useful to express the results in terms of the reduced coefficients  $\kappa_{G,j}$ . For the family of bipyramid graphs  $B_n$ , Eqs. (6.1)-(6.4) of Ref. [1] are equivalently expressed as Eq. (3.16), with  $n(B_m) = m + 2$ , which is well-defined for  $m \geq m_{min} = 3$ , with the reduced coefficients

$$\kappa_{B,1} = \kappa_{B,2} = \kappa_{B,3} = 1 . \quad (4.1)$$

For this family,  $r(B_\infty) = (-1 + \sqrt{5})/2 = 0.6180..$  and, as a special case of (3.18),  $a_B = 1$ . Here and below we will index the members of the various families of planar triangulations as  $G_{pt,m}$ , equivalent to the notation  $G_{pt,n}$  that we used for some families in [1] via the relation (2.1).

#### V. THE FAMILY $H_m$

Here we remark on another family of planar triangulations  $H_m$  [1], which is well-defined for  $m \geq m_{min} = 3$  and has  $n(H_m) = m + 5$ . In Fig. 1 we show the lowest member of the family,  $H_3$ . The next higher member,  $H_4$ , is constructed by adding a vertex and associated



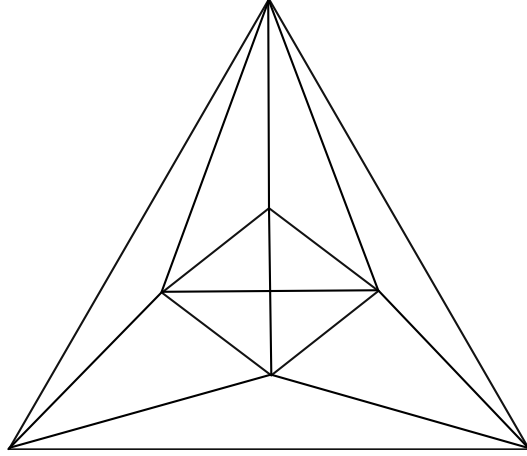


FIG. 1: Graph  $H_3$ .

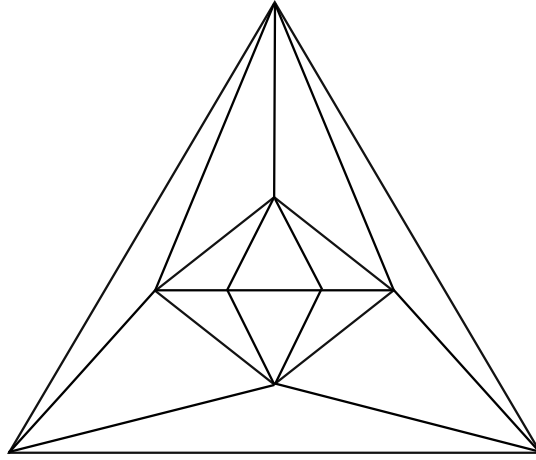


FIG. 2: Graph  $H_4$ .

edges in the central diamond-like subgraph, as shown in Fig. 2, and so forth for higher members. We have given the chromatic polynomial for this family in Ref. [1] and have analyzed its properties there. In our present notation,  $P(H_m, q)$  has the form (3.16) with

$$\kappa_{H,1} = (q - 3)^3 , \quad (5.1)$$

$$\kappa_{H,2} = q^3 - 9q^2 + 30q - 35 , \quad (5.2)$$

and

$$\kappa_{H,3} = -(q - 3)(q - 5) . \quad (5.3)$$

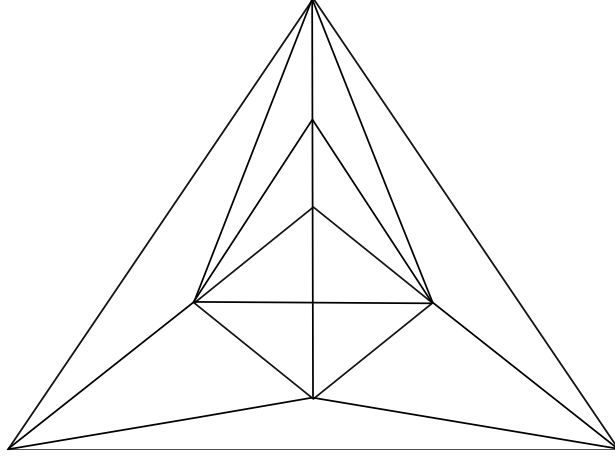


FIG. 3: Graph  $L_4$ .

For this family,  $r(H_\infty) = (7 - 3\sqrt{5})/2 = 0.145898..$  [1].

## VI. THE FAMILY $L_m$

For comparative purposes, it is useful to study another family of planar triangulations with chromatic polynomials of the multi-term form (2.9). Here we denote this family as  $L_m$ . It is well-defined for  $m \geq m_{min} = 3$  and has  $n(L_m) = m + 5$ . The lowest member of this family,  $L_3$ , is the same as  $H_3$ , shown in Fig. 1. The next higher member,  $L_4$ , is constructed by adding a vertex and associated edges in the central “diamond”, as shown in Fig. 3, and so forth for higher members.

For this family of planar triangulations we calculate the chromatic polynomial  $P(L_m, q)$  to be of the form (3.16) with

$$\kappa_{L,1} = (q - 2)(q - 3)^2 , \quad (6.1)$$

$$\kappa_{L,2} = q^3 - 9q^2 + 29q - 32 , \quad (6.2)$$

(equal to  $\lambda_{TC}$ ) and

$$\kappa_{L,3} = 2(q - 3) . \quad (6.3)$$

$P(L_m, q)$  contains the factor  $P(K_4, q)$  and has  $\chi(L_m) = 4$ .

Evaluating  $P(L_m, q)$  at  $q = \tau + 1$ , we find

$$P(L_m, \tau + 1) = (-2 + \sqrt{5}) \left[ (\tau - 1)^m + 2(\tau - 2)^m \right] . \quad (6.4)$$

Consequently,

$$r(L_m) = (-2 + \sqrt{5}) \left[ 1 + 2 \left( \frac{1 - \sqrt{5}}{2} \right)^m \right]. \quad (6.5)$$

As before, since  $|(1 - \sqrt{5})/2| < 1$ , the second term in Eq. (6.5) vanishes (exponentially fast) as  $m \rightarrow \infty$ , so

$$r(L_\infty) = -2 + \sqrt{5} = 0.236068 \quad (6.6)$$

(to the indicated accuracy) and, as a special case of Eq. (3.18),  $a_L = 1$ .

We proved in general above that for any family of planar triangulations  $G_{pt,m}$  with chromatic polynomials  $P(G_{pt,m}, q)$  of the form (3.16),  $P(G_{pt,m}, q)$  has a zero that approaches  $\tau + 1$  as  $m \rightarrow \infty$ . The families  $B_m$ ,  $H_m$ , and  $L_m$  (as well as others to be discussed below) illustrate this general result. In the present case, for odd  $m$  and hence even  $n = m + 5$ , this zero of  $P(L_m, q)$  is slightly less than  $q = \tau + 1$ , while for even  $m$  and hence odd  $n$ , the nearby zero is slightly greater than  $\tau + 1$ . We list these zeros in Table I for  $m$  from 4 to 15 (i.e.,  $n$  from 9 to 20).

If and only if  $m$  is odd, i.e.,  $n$  is even,  $P(L_m, q)$  has another real zero somewhat larger than 3, which decreases monotonically toward 3 from above as  $m \rightarrow \infty$ . As examples, for  $n = 8, 12, 16, 20$ , and 24, this zero occurs at approximately  $q = 3.61, 3.37, 3.25, 3.19$ , and 3.16, respectively.

## VII. TWO-PARAMETER FAMILIES OF PLANAR TRIANGULATIONS,

$$G_{pt,m_1,m_2}$$

In this section we introduce a substantial generalization to a two-parameter family  $G_{pt,m_1,m_2}$  of planar triangulations involving the three  $\lambda_j$ 's in (3.1), with a chromatic polynomial of the form

$$P(G_{pt,m_1,m_2}, q) = \sum_{i_1=1}^3 \sum_{i_2=1}^3 c_{G,i_1 i_2} \lambda_{i_1}^{m_1} \lambda_{i_2}^{m_2}. \quad (7.1)$$

Explicitly,

$$\begin{aligned} P(G_{pt,m_1,m_2}, q) = & c_{G,11}(q-2)^{m_1+m_2} + c_{G,22}(q-3)^{m_1+m_2} + c_{G,33}(-1)^{m_1+m_2} \\ & + c_{G,12}(q-2)^{m_1}(q-3)^{m_2} + c_{G,21}(q-3)^{m_1}(q-2)^{m_2} \\ & + c_{G,13}(q-2)^{m_1}(-1)^{m_2} + c_{G,31}(-1)^{m_1}(q-2)^{m_2} \end{aligned}$$

TABLE I: Location of zero  $q_z$  of  $P(L_m, q)$  closest to  $\tau + 1$ , as a function of the number of vertices,  $n = m + 5$ .

Notation  $a\text{e-}\nu$  means  $a \times 10^{-\nu}$  here and in tables below.

$n$	$q_z$	$q_z - (\tau + 1)$
9	2.630048	0.01201
10	2.614750	-0.003284
11	2.621594	0.003560
12	2.616447	-0.0015865
13	2.619250	0.001216
14	2.617375	-0.6588e-3
15	2.618477	0.4432e-3
16	2.617774	-2.598e-4
17	2.618200	1.661e-4
18	2.6179335	-1.005e-4
19	2.618097	0.6294e-4
20	2.617995	-3.8574e-5

$$+ c_{G,23}(q-3)^{m_1}(-1)^{m_2} + c_{G,32}(-1)^{m_1}(q-3)^{m_2} . \quad (7.2)$$

(Below we shall often take  $i_1 = i$ ,  $i_2 = j$  to simplify the notation.) Clearly, if one keeps one of the indices  $m_1$  or  $m_2$  fixed and varies the other, this defines an infinite set of one-parameter families of planar triangulations. For specific families  $G_{pt,m_1,m_2}$  we will show how the general structure (7.1) reduces, in such cases, to the form (3.16) considered above for a class of one-parameter planar triangulations, with  $m$  being equal to the variable index, up to an appropriate integer shift.

As before for the one-parameter families, an equivalent way to obtain the  $P(G_{pt,m_1,m_2}, q)$  is via a Taylor series expansion, in the auxiliary variables  $x_1$  and  $x_2$ , of a generating function  $\Gamma(G_{pt}, q, x_1, x_2)$ . Equivalent to both of these is the property that  $P(G_{pt,m_1,m_2}, q)$  satisfies a two-dimensional recursion relation, for  $m_1 \geq (m_1)_{min} + 3$  and  $m_2 \geq (m_2)_{min} + 3$ ,

$$P(G_{pt,m_1,m_2}, q) + \sum_{i_1=1}^3 \sum_{i_2=1}^3 b_{G_{pt},i_1,i_2} P(G_{pt,m_1-i_1,m_2-i_2}, q) = 0 . \quad (7.3)$$

The coefficients in this recursion relation are given by

$$1 + \sum_{i_1=1}^3 \sum_{i_2=1}^3 b_{G_{pt}, i_1 i_2} x_1^{i_1} x_2^{i_2} = \left[ \prod_{i_1=1}^3 (1 - \lambda_{i_1} x_1) \right] \left[ \prod_{i_2=1}^3 (1 - \lambda_{i_2} x_2) \right]. \quad (7.4)$$

Note that they satisfy the symmetry property

$$b_{G_{pt}, i_1 i_2} = b_{G_{pt}, i_2 i_1}. \quad (7.5)$$

We first derive a number of restrictions on the coefficients  $c_{G, i_1 i_2}$ . As is true of any triangulation,  $P(G_{pt, m_1, m_2}, q) = 0$  for  $q = 0$ ,  $q = 1$ , and  $q = 2$ . The evaluation  $P(G_{pt, m_1, m_2}, 0) = 0$  reads

$$\begin{aligned} & c_{G, 11}(-2)^{m_1+m_2} + c_{G, 22}(-3)^{m_1+m_2} + c_{G, 33}(-1)^{m_1+m_2} \\ & + c_{G, 12}(-2)^{m_1}(-3)^{m_2} + c_{G, 21}(-3)^{m_1}(-2)^{m_2} \\ & + c_{G, 13}(-2)^{m_1}(-1)^{m_2} + c_{G, 31}(-1)^{m_1}(-2)^{m_2} \\ & + c_{G, 23}(-3)^{m_1}(-1)^{m_2} + c_{G, 32}(-1)^{m_1}(-3)^{m_2} = 0 \quad \text{at } q = 0. \end{aligned} \quad (7.6)$$

Since this equation applies for arbitrary  $m_1$  and  $m_2$  in their respective ranges, it implies that  $c_{G, i_1 i_2} = 0$  for all  $i_1, i_2$  at  $q = 0$  and hence that

$$c_{G, i_1 i_2} \quad \text{contains the factor } q \quad \forall i_1, i_2. \quad (7.7)$$

It will often be convenient to extract this common factor, via the definition

$$\bar{c}_{G, i_1 i_2} = \frac{c_{G, i_1 i_2}}{q}. \quad (7.8)$$

The evaluation  $P(G_{pt, m_1, m_2}, 1) = 0$  reads

$$\begin{aligned} & [c_{G, 11} + c_{G, 33} + c_{G, 13} + c_{G, 31}](-1)^{m_1+m_2} + c_{G, 22}(-2)^{m_1+m_2} \\ & + [c_{G, 12} + c_{G, 32}](-1)^{m_1}(-2)^{m_2} \\ & + [c_{G, 21} + c_{G, 23}](-2)^{m_1}(-1)^{m_2} = 0 \quad \text{at } q = 1. \end{aligned} \quad (7.9)$$

Since this equation applies for arbitrary  $m_1$  and  $m_2$ , it implies the conditions

$$c_{G, 11} + c_{G, 33} + c_{G, 13} + c_{G, 31} = 0, \quad c_{G, 22} = 0,$$

$$c_{G,12} + c_{G,32} = 0, \quad c_{G,21} + c_{G,23} = 0 \quad \text{at } q = 1. \quad (7.10)$$

In particular, this implies that

$$c_{G,22} \quad \text{contains the factor } q - 1. \quad (7.11)$$

The evaluation  $P(G_{pt,m_1,m_2}, 2) = 0$  reads

$$\begin{aligned} & c_{G,11} 0^{m_1+m_2} + [c_{G,22} + c_{G,33} + c_{G,23} + c_{G,32}](-1)^{m_1+m_2} \\ & + [c_{G,12} + c_{G,13}]0^{m_1}(-1)^{m_2} + [c_{G,21} + c_{G,31}](-1)^{m_1}0^{m_2} = 0 \quad \text{at } q = 2. \end{aligned} \quad (7.12)$$

Since this equation applies for arbitrary  $m_1$  and  $m_2$ , including  $m_1 = m_2 = 0$ , it implies the conditions

$$\begin{aligned} & c_{G,11} = 0, \quad c_{G,22} + c_{G,33} + c_{G,23} + c_{G,32} = 0, \\ & c_{G,12} + c_{G,13} = 0, \quad c_{G,21} + c_{G,31} = 0 \quad \text{at } q = 2. \end{aligned} \quad (7.13)$$

In particular, this implies that

$$c_{G,11} \quad \text{contains the factor } q - 2. \quad (7.14)$$

For families  $G_{pt,m_1,m_2}$  with  $\chi(G_{pt,m_1,m_2}) = 4$  for certain values of  $m_1$  and  $m_2$ , further conditions hold, as we shall discuss below.

We next derive some further restrictions on the coefficients  $c_{G,i_1 i_2}$  from the condition that  $P(G_{pt,m_1,m_2}, q)$  must obey the Tutte upper bound when evaluated at  $q = \tau + 1$ . We consider families such that

$$n(G_{pt,m_1,m_2}) = m_1 + m_2 + \beta \quad (7.15)$$

Carrying out this evaluation and calculating the ratio  $r(G_{pt,m_1,m_2})$ , we have

$$\begin{aligned} r(G_{pt,m_1,m_2}) &= (\tau - 1)^{5-\beta} \left| c_{G,11} + c_{G,22} \left( \frac{\tau - 2}{\tau - 1} \right)^{m_1+m_2} + c_{G,33} \left( \frac{-1}{\tau - 1} \right)^{m_1+m_2} \right. \\ &+ c_{G,12} \left( \frac{\tau - 2}{\tau - 1} \right)^{m_2} + c_{G,21} \left( \frac{\tau - 2}{\tau - 1} \right)^{m_1} \\ &+ c_{G,13} \left( \frac{-1}{\tau - 1} \right)^{m_2} + c_{G,31} \left( \frac{-1}{\tau - 1} \right)^{m_1} \\ &\left. + c_{G,23} \frac{(\tau - 2)^{m_1}(-1)^{m_2}}{(\tau - 1)^{m_1+m_2}} + c_{G,32} \frac{(-1)^{m_1}(\tau - 2)^{m_2}}{(\tau - 1)^{m_1+m_2}} \right|. \end{aligned} \quad (7.16)$$

The condition that  $r(G_{pt,m_1,m_2}) \leq 1$  for arbitrary  $m_1$  and  $m_2$  implies that

$$c_{G,33} = c_{G,13} = c_{G,31} = c_{G,23} = c_{G,32} = 0 \quad \text{at } q = \tau + 1. \quad (7.17)$$

By the same argument that we used above for the analysis of the coefficients of chromatic polynomials of one-parameter planar triangulation graphs, (7.17) implies that

$$c_{G,i_1 i_2} \quad \text{contains the factor } q^2 - 3q + 1 \quad \text{if } i_1 = 3 \text{ or } i_2 = 3. \quad (7.18)$$

By taking either  $m_1 \rightarrow \infty$  or  $m_2 \rightarrow \infty$ , and requiring that the resultant ratio  $r(G_{pt,\infty,m_2})$  or  $r(G_{pt,m_1,\infty})$  must obey the Tutte upper bound, one deduces the inequality

$$(\tau - 1)^{5-\beta} |c_{G,11}| < 1 \quad \text{at } q = \tau + 1. \quad (7.19)$$

We next generalize our result on a real chromatic zero that approaches  $q = \tau + 1$  for one-parameter planar triangulations to these two-parameter planar triangulations with  $P(G_{pt,m_1,m_2}, q)$  of the form (7.1). As in the  $p = 1$  case, we assume that  $q$  is a real number in the interval  $[q_w, 3)$ . We will actually obtain two results, corresponding to  $m_1 \rightarrow \infty$  for fixed  $m_2$  and  $m_2 \rightarrow \infty$  for fixed  $m_1$ . For the first of these limits, the six terms proportional to  $c_{G,11}$ ,  $c_{G,22}$ ,  $c_{G,12}$ ,  $c_{G,21}$ ,  $c_{G,13}$ , and  $c_{G,23}$  all vanish (exponentially rapidly), so that

$$P(G_{pt,m_1,m_2}, q) \sim c_{G,33}(-1)^{m_1+m_2} + c_{G,31}(-1)^{m_1}(q-2)^{m_2} + c_{G,32}(-1)^{m_1}(q-3)^{m_2} \quad \text{for } m_1 \rightarrow \infty \quad (7.20)$$

But we have shown above in (7.18) that  $c_{G,i_1 i_2}$  contains the factor  $q^2 - 3q + 1$  if  $i_1 = 3$  or  $i_2 = 3$ . Since this factor vanishes at  $q = \tau + 1$  in this interval  $[q_w, 3)$ , it follows that for sufficiently large  $m_1$ ,  $P(G_{pt,m_1,m_2}, q)$  has a real zero that approaches  $\tau + 1$ . With obvious changes, a corresponding argument shows that for sufficiently large  $m_2$  and fixed  $m_1$ ,  $P(G_{pt,m_1,m_2}, q)$  has a real zero that approaches  $\tau + 1$ . Clearly, the result also holds if both  $m_1$  and  $m_2$  get large.

## VIII. GENERAL FORM OF $P(G_{pt,\vec{m}}, q)$

The generalization of our structural results for two-parameter families of planar triangulations,  $G_{pt,m_1,m_2}$  to  $p$ -parameter families is as follows. Let  $G_{pt,\vec{m}}$  be a family of planar triangulation graphs involving the three  $\lambda_j$ 's in (3.1) and depending on the  $p$  parameters  $\vec{m} = (m_1, \dots, m_p)$  taking on integer values in the ranges  $m_i \geq (m_i)_{min}$ ,  $i = 1, \dots, p$ . Then

$$P(G_{pt,m_1,\dots,m_p}, q) = \sum_{i_1=1}^3 \cdots \sum_{i_p=1}^3 c_{G_{pt,i_1\dots i_p}} \left[ \prod_{\ell=1}^p \lambda_{i_\ell}^{m_\ell} \right]. \quad (8.1)$$

In general, there are  $3^p$  terms involving products of the  $\lambda$ 's (multiplied by respective coefficients) in this sum.

This general form is of considerable interest. It shows that one can carry out a  $p$ -fold sequence of edge proliferations, each of which involves arbitrarily many additional edges, as indexed by the parameters  $m_1, \dots, m_p$ , with the chromatic polynomial  $P(G_{pt, \vec{m}}, q)$  still retaining the rather simple form (8.1) with the same set of three  $\lambda_i$ 's given in (3.1). This is a much simpler situation than that in previous calculations of chromatic polynomials for multiparameter families of graphs. For example, in [28], Tsai and one of the present authors calculated the chromatic polynomial  $P(G_{e_1, e_2, e_g, m}, q)$  for a certain four-parameter family of cyclic chain graphs in which each subgraph on the chain has  $e_1$  edges above, and  $e_2$  edges below, the main line, with  $e_g$  edges between the subgraphs, and  $m$  subgraphs in all. Although the number  $N_\lambda$  of  $\lambda_{G,j}$ 's for this family has the fixed value of 2, the  $\lambda_{G,j}$ 's have functional forms that depend on the parameters  $e_1$ ,  $e_2$ , and  $e_g$  (as is the case for the full Potts model partition function [29]). This property was also found to be true for (i) the chromatic polynomials  $P((Ch)_{k,m,cyc}, q)$  and  $P((Ch)_{k,m,Mb}, q)$  of cyclic (*cyc.*) and Möbius (*Mb.*) strips depending on a homeomorphic expansion parameter  $k$  and the strip length,  $m$ , where  $N_\lambda = 4$  and three of the  $\lambda_{Ch,j}$ 's depended on  $k$  [30]; and (ii)  $P(H_{k,r}, q)$  for a family of “hammock” graphs  $H_{k,r}$  with  $r$  “ropes” (linear sets of edges) joining two end vertices, with each rope having  $k$  “knots” (vertices), where again the  $N_\lambda = 2$  terms  $\lambda_{H,j}$  depended on  $k$  and  $r$  [21, 22]. The remarkable simplicity of the form (8.1) is a result of the restrictive property that  $G_{pt, \vec{m}}$  is a planar triangulation. We know that this simple behavior does not obtain even for the lowest case of one-parameter families for planar near-triangulations, from the explicit calculation the chromatic polynomials for free strips of the triangular lattice of length  $m$  and width  $L_y = 2, 3$  [31] (which are near-triangulations), where it was found that the  $\lambda$ 's changed with increasing width [31]. (Here a near-triangulation is defined as a graph such that all faces except one are triangles.) We also know that it does not hold for nonplanar triangulations, from explicit calculations of chromatic polynomials for the  $L_y = 2$  [32],  $L_y = 3$  [33], and  $L_y = 4$  [34] strips of the triangular lattice with doubly periodic (toroidal) boundary conditions. Thus, chromatic polynomials of multiparameter families of planar triangulation are especially amenable to exact analytic treatment.

The  $P(G_{pt, m_1, \dots, m_p}, q)$  satisfy a  $p$ -dimensional recursion relation, for  $m_\ell \geq (m_\ell)_{min} + 3$ ,  $\ell = 1, \dots, p$ , namely

$$P(G_{pt, m_1, \dots, m_p}, q) + \sum_{i_1=1}^3 \cdots \sum_{i_p=1}^3 b_{G_{pt}, i_1 \dots i_p} P(G_{pt, m_1 - i_1, \dots, m_p - i_p}, q) = 0 \quad (8.2)$$



where the  $b_{G_{pt}, i_1 \dots i_p}$  are given by

$$1 + \sum_{i_1=1}^3 \cdots \sum_{i_p=1}^3 b_{G_{pt}, i_1 \dots i_p} \left( \prod_{s=1}^p x_s^{i_s} \right) = \prod_{\ell=1}^p \left[ \prod_{i=1}^3 (1 - \lambda_i x_\ell) \right]. \quad (8.3)$$

Using the same methods as for  $p = 2$ , it is straightforward to generalize our results to this case, including (i) the conditions on the coefficients  $c_{G_{pt}, \vec{i}}$  (where  $\vec{i} \equiv (i_1 \dots i_p)$ ) derived from the evaluations  $P(G_{pt, \vec{m}}, q) = 0$  for  $q = 0, 1, 2$  and the Tutte upper bound at  $q = \tau + 1$ , and (ii) the results for  $r(G_{pt, \vec{m}})$  and its limits as one or more of the  $m_i \rightarrow \infty$ . Clearly, our result on a real zero in the interval  $[q_w, 3)$  that approaches  $\tau + 1$  also generalizes to this case of families  $G_{pt, \vec{m}}$  with  $p \geq 3$ .

As in the  $p = 2$  case, if one holds all but one of the  $m_1, \dots, m_p$  fixed and allows one to vary, then the general form (8.1) reduces to (3.16) with  $m$  being equal to the variable parameter, up to an appropriate integer shift.

## IX. THE TWO-PARAMETER FAMILY $D_{m_1, m_2}$

We proceed to analyze our first explicit two-parameter family of planar triangulations, denoted  $D_{m_1, m_2}$  (where  $D$  stands for the proliferation of a double set of edges). To explain the general method of construction of this family, we show in Fig. 4 the lowest member of the series, namely the graph  $D_{0,0}$ . We now add  $m_1$  inner edges joining the uppermost vertex to the upper horizontal edge (thereby producing several such upper horizontal edges) and, separately, add  $m_2$  inner edges joining the central vertex to the lower horizontal edge (thereby producing several such lower horizontal edges), with corresponding edges connecting to the lower central vertex. Thus, in the  $D_{m_1, m_2}$  graph, the uppermost vertex has degree  $6 + m_1$ , the central vertex has degree  $m_1 + m_2 + 4$ , and the lower central vertex has degree  $4 + m_2$ . To illustrate this, we show the graphs  $D_{1,2}$ ,  $D_{2,2}$  in Figs. 5 and 6.

From inspection of these graphs, it is evident how to construct  $D_{m_1, m_2}$  graphs with higher values of  $m_1$  and  $m_2$ . The number of vertices in the graph  $D_{m_1, m_2}$  is

$$n(D_{m_1, m_2}) = m_1 + m_2 + 9. \quad (9.1)$$

For the chromatic number, we find that (i) if  $m_2$  is odd, then  $\chi(D_{m_1, m_2}) = 4$ , and (ii) if  $m_2$  is even, then  $\chi(D_{m_1, m_2}) = 3$  if  $m_1$  is even and  $\chi(D_{m_1, m_2}) = 4$  if  $m_1$  is odd. That is, denoting even as  $e$  and odd as  $o$ ,  $\chi(D_{m_1, m_2}) = 3$  for  $(m_1, m_2) = (e, e)$  and  $\chi(D_{m_1, m_2}) = 4$  for  $(m_1, m_2) = (e, o)$ ,  $(o, o)$ , and  $(o, e)$ . The chromatic polynomials for the  $D_{m_1, m_2}$  with  $\chi = 4$  contain a factor  $P(K_4, q) = q(q-1)(q-2)(q-3)$  (and some contain an additional

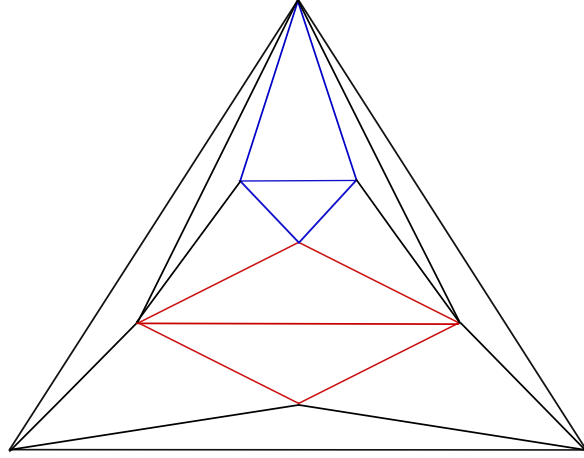


FIG. 4: Graph  $D_{0,0}$ .

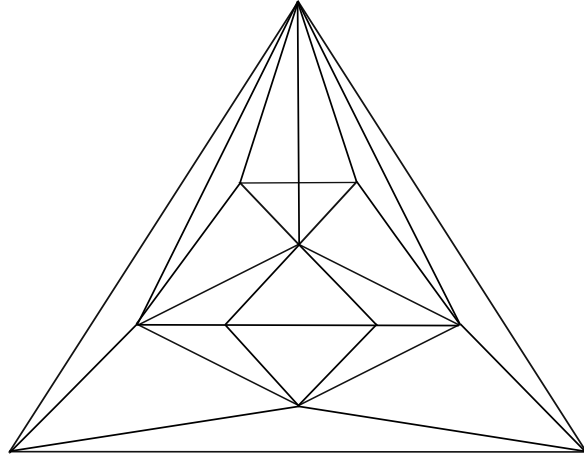


FIG. 5: Graph  $D_{1,2}$ .

factor of  $(q - 3)$ ). Of course, all chromatic polynomials of triangulations have the factor  $P(K_3, q) = q(q - 1)(q - 2)$ .

By means of an iterative use of the deletion-contraction relation, we have calculated the chromatic polynomial  $P(D_{m_1, m_2}, q)$  for an arbitrary graph in this general two-parameter  $D_{m_1, m_2}$  family. We find that  $P(D_{m_1, m_2}, q)$  has the form (7.1) with coefficients  $c_{D, ij}$  that are rational functions of  $q$ . With the definition (7.8), we calculate

$$\bar{c}_{D, 11} = \frac{(q - 2)^7}{q - 1} , \quad (9.2)$$

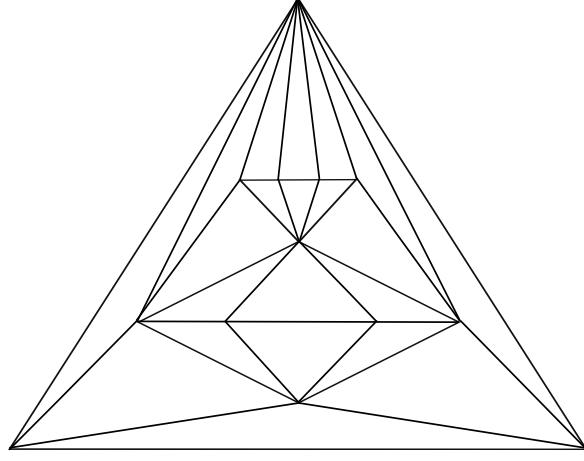


FIG. 6: Graph  $D_{2,2}$ .

$$\bar{c}_{D,22} = \frac{(q-1)(q-3)^5(q^3 - 9q^2 + 30q - 35)}{q-2}, \quad (9.3)$$

$$\bar{c}_{D,33} = \frac{(q^5 - 11q^4 + 46q^3 - 88q^2 + 74q - 23)(q^2 - 3q + 1)}{(q-1)(q-2)}, \quad (9.4)$$

$$\bar{c}_{D,12} = (q-2)^3(q-3)^4, \quad (9.5)$$

$$\bar{c}_{D,21} = (q-2)(q-3)^6, \quad (9.6)$$

$$\bar{c}_{D,13} = \frac{(q-2)^3(q^2 - 3q + 1)}{q-1}, \quad (9.7)$$

$$\bar{c}_{D,31} = \frac{(q-2)(q^2 - 3q + 1)}{q-1}, \quad (9.8)$$

$$\bar{c}_{D,23} = -\frac{(q-3)^4(q-5)(q^2 - 3q + 1)}{q-2}, \quad (9.9)$$

and

$$\bar{c}_{D,32} = -\frac{(q-3)^2(q-5)(q^2 - 3q + 1)}{q-2}. \quad (9.10)$$

It should be noted that the poles at  $q = 1$  and  $q = 2$  in certain of these  $c_{D,ij}$  coefficients are cancelled in the actual evaluation of  $P(D_{m_1, m_2}, q)$ , which is, as it must be, a polynomial in  $q$ . Furthermore, not only are these poles cancelled, but also the resultant  $P(D_{m_1, m_2}, q)$  vanishes at  $q = 0$ ,  $q = 1$ , and  $q = 2$ . It is easily verified that the coefficients  $c_{D,ij}$  satisfy the requisite conditions (7.10) and (7.13) for these zeros to occur.

Furthermore, since  $c_{D,11} = c_{D,33} = c_{D,13} = c_{D,31} = 3/2$  with the other  $c_{D,ij} = 0$  at  $q = 3$ , it follows that

$$P(D_{m_1, m_2}, 3) = \frac{3}{2} \left[ 1 + (-1)^{m_1+m_2} + (-1)^{m_2} + (-1)^{m_1} \right]. \quad (9.11)$$

This vanishes for  $(m_1, m_2) = (e, o)$ ,  $(o, o)$ , and  $(o, e)$ , and is nonvanishing for  $(m_1, m_2) = (e, e)$  in our notation above, in agreement with our result on the chromatic number  $\chi(D_{m_1, m_2})$ .

We proceed to discuss the evaluation of the chromatic polynomial  $P(D_{m_1, m_2}, q)$  at  $q = \tau + 1$  and the comparison with the Tutte upper bound. We have

$$\begin{aligned} P(D_{m_1, m_2}, \tau + 1) &= (9 - 4\sqrt{5})(\tau - 1)^{m_1+m_2} + \left( \frac{445 - 199\sqrt{5}}{2} \right) (\tau - 2)^{m_1+m_2} \\ &+ \left( \frac{-38 + 17\sqrt{5}}{2} \right) (\tau - 1)^{m_1} (\tau - 2)^{m_2} \\ &+ \left( \frac{-199 + 89\sqrt{5}}{2} \right) (\tau - 2)^{m_1} (\tau - 1)^{m_2} \end{aligned} \quad (9.12)$$

Comparing this with the Tutte upper bound  $(\tau - 1)^{m_1+m_2+4}$ , we have

$$\begin{aligned} r(D_{m_1, m_2}) &= \frac{3 - \sqrt{5}}{2} + \left( \frac{65 - 29\sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^{m_1+m_2} \\ &+ \left( \frac{-11 + 5\sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^{m_2} \\ &+ \left( \frac{-29 + 13\sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^{m_1}. \end{aligned} \quad (9.13)$$

We list a number of values of  $r(D_{m_1, m_2})$  in Table II. One may investigate the behavior of  $r(D_{m_1, m_2})$  as  $m_1 \rightarrow \infty$  for fixed  $m_2$  and as  $m_2 \rightarrow \infty$  for fixed  $m_1$ . Because the quantity  $(1 - \sqrt{5})/2$  that is raised to the powers indicated in (9.13) is negative, it follows that, if one keeps  $m_2$  fixed and increases  $m_1$ , then  $r(D_{m_1, m_2})$  does not approach  $r(D_{\infty, m_2})$  monotonically, although the members of the subsequences  $r(D_{m_1, m_2})$  with even (odd)  $m_1$  approach  $r(D_{\infty, m_2})$  monotonically from above (below), respectively. Similarly, if one keeps  $m_1$  fixed and increases  $m_2$ , then  $r(D_{m_1, m_2})$  does not approach  $r(D_{m_1, \infty})$  monotonically, although the members of the subsequences  $r(D_{m_1, m_2})$  with even (odd)  $m_2$  approach  $r(D_{m_1, \infty})$  monotonically from above (below), respectively. The results are

$$r(D_{\infty, m_2}) \equiv \lim_{m_1 \rightarrow \infty} r(D_{m_1, m_2}) = \frac{3 - \sqrt{5}}{2} + \left( \frac{-11 + 5\sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^{m_2} \quad (9.14)$$

$$r(D_{m_1, \infty}) \equiv \lim_{m_2 \rightarrow \infty} r(D_{m_1, m_2}) = \frac{3 - \sqrt{5}}{2} + \left( \frac{-29 + 13\sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^{m_1} \quad (9.15)$$

In the limit where one takes both  $m_1$  and  $m_2$  to  $\infty$ , one has

$$\begin{aligned} r(D_{\infty,\infty}) &= \lim_{m_1 \rightarrow \infty} \lim_{m_2 \rightarrow \infty} r(D_{m_1,m_2}) = \lim_{m_2 \rightarrow \infty} \lim_{m_1 \rightarrow \infty} r(D_{m_1,m_2}) \\ &= \frac{3 - \sqrt{5}}{2} = 2 - \tau = 0.381966... \end{aligned} \quad (9.16)$$

As  $m_2$  increases from 0 to  $\infty$ ,  $r(D_{\infty,m_2})$  decreases (non-monotonically) from the value

$$r(D_{\infty,0}) = -4 + 2\sqrt{5} = 0.4721359... \quad (9.17)$$

to the value in (9.16), and as  $m_1$  increases from 0 to  $\infty$ ,  $r(D_{m_1,\infty})$  decreases (non-monotonically) from the value

$$r(D_{0,\infty}) = -13 + 6\sqrt{5} = 0.4164078... \quad (9.18)$$

to the value in (9.16). As a consequence of the relation (9.34) (see below), it follows that

$$r(D_{\infty,k+2}) = r(D_{k,\infty}) \quad (9.19)$$

In general, the maximal value of  $r(D_{m_1,m_2})$  occurs for the member of the  $D_{m_1,m_2}$  family with the minimal values of  $m_1$  and  $m_2$ , namely for  $D_{0,0}$ . This property is similar to the property that the maximum value of  $r(G_{pt})$  for all planar triangulations  $G_{pt}$  occurs for the  $G_{pt}$  with the minimum number of vertices, namely the single triangle,  $K_3$ .

We next show that our general form for  $P(D_{m_1,m_2}, q)$  reduces to (3.16) when either  $m_2$  is held fixed and  $m_1$  varies, or vice versa. If we keep  $m_2$  fixed and vary  $m_1$ , then we can write  $P(D_{m_1,m_2}, q)$  as

$$P(D_{m_1,m_2}, q) = \sum_{i=1}^3 \left[ \sum_{j=1}^3 c_{D,ij} \lambda_j^{m_2} \right] \lambda_i^{m_1} . \quad (9.20)$$

The sum  $\sum_{j=1}^3 c_{D,ij} \lambda_j^{m_2}$  contains a factor  $\lambda_i^4$ , which we combine with the  $\lambda_i^{m_1}$ , to make  $\lambda_i^m$ , where

$$m = m_1 + 4 . \quad (9.21)$$

This shows that  $P(D_{m_1,m_2}, q)$  has the form (3.16) with  $m$  given by (9.21); explicitly,

$$P(D_{m-4,m_2}, q) = \sum_{i=1}^3 c_{D_{m_2(\ell)},i} \lambda_i^m , \quad (9.22)$$

where

$$c_{D_{m_2(\ell)},i} = \lambda_i^{-4} \sum_{j=1}^3 c_{D,ij} \lambda_j^{m_2} . \quad (9.23)$$

TABLE II: Values of the ratio  $r(D_{m_1, m_2})$ . The rows and columns list  $m_1$  and  $m_2$ , respectively, so that, for example,  $r(D_{1,2})$  is the entry 0.3769.

$m_1, m_2$	0	1	2	3	4	5	6	$\infty$
0	0.5836	0.3131	0.4803	0.3769	0.4408	0.4013	0.4257	0.4164
1	0.4033	0.3344	0.3769	0.3506	0.3669	0.3568	0.3631	0.3607
2	0.5147	0.3212	0.4408	0.3669	0.4126	0.3843	0.4018	0.3951
3	0.4458	0.3293	0.4013	0.3568	0.3843	0.3673	0.3778	0.3738
4	0.4884	0.3243	0.4257	0.36305	0.4018	0.3778	0.3926	0.3870
5	0.4621	0.3274	0.4106	0.3592	0.3910	0.3714	0.3835	0.3789
6	0.4783	0.3255	0.4200	0.3616	0.3977	0.3754	0.38915	0.3839
$\infty$	0.4721	0.3262	0.4164	0.3607	0.3951	0.3738	0.3870	0.3820

Here, since these coefficients depend only on  $m_2$ , and not on  $m_1$ , we have introduced the notation  $D_{m_2(\ell)}$  to refer to the entire family  $D_{m_1, m_2}$  with fixed  $m_2$  and variable  $m_1$ , where  $\ell$  indicates that  $m_2$  describes the edge proliferation in the lower part of the graph. Expressing (3.13)-(3.15) in our notation (and suppressing the  $q$  arguments), we have

$$c_{D_{m_2(\ell)}, 1} = q \kappa_{D_{m_2(\ell)}, 1} , \quad (9.24)$$

$$c_{D_{m_2(\ell)}, 2} = q(q-1) \kappa_{D_{m_2(\ell)}, 2} , \quad (9.25)$$

and

$$c_{D_{m_2(\ell)}, 3} = q(q^2 - 3q + 1) \kappa_{D_{m_2(\ell)}, 3} . \quad (9.26)$$

Similarly, if we keep  $m_1$  fixed and vary  $m_2$ , then we can write  $P(D_{m_1, m_2}, q)$  as

$$P(D_{m_1, m_2}, q) = \sum_{j=1}^3 \left[ \sum_{i=1}^3 c_{D, ij} \lambda_i^{m_1} \right] \lambda_j^{m_2} . \quad (9.27)$$

The sum  $\sum_{i=1}^3 c_{D, ij} \lambda_i^{m_1}$  contains a factor  $\lambda_j^2$ , which we combine with the  $\lambda_j^{m_2}$ , to make  $\lambda_j^m$ , where, for this one-parameter reduction,

$$m = m_2 + 2 . \quad (9.28)$$

This shows that  $P(D_{m_1, m_2}, q)$  has the form (3.16) with  $m$  given by (9.28); explicitly,

$$P(D_{m_1, m_2}, q) = \sum_{j=1}^3 c_{D_{m_1(u)}, j} \lambda_j^m, \quad (9.29)$$

where

$$c_{D_{m_1(u)}, j} = \lambda_j^{-2} \sum_{i=1}^3 c_{D, ij} \lambda_i^{m_1}. \quad (9.30)$$

Here again, since these coefficients depend only on  $m_1$ , and not on  $m_2$ , we have introduced the notation  $D_{m_1(u)}$  to refer to the entire family  $D_{m_1, m_2}$  with fixed  $m_1$  and variable  $m_2$ , where  $u$  indicates that  $m_1$  describes the edge proliferation in the upper part of the graph. As before, we write

$$c_{D_{m_1(u)}, 1} = q \kappa_{D_{m_1(u)}, 1}, \quad (9.31)$$

$$c_{D_{m_1(u)}, 2} = q(q-1) \kappa_{D_{m_1(u)}, 2}, \quad (9.32)$$

and

$$c_{D_{m_1(u)}, 3} = q(q^2 - 3q + 1) \kappa_{D_{m_1(u)}, 3}. \quad (9.33)$$

We find that

$$c_{D_{k+2(\ell)}, i} = c_{D_{k(u)}, i} \quad \text{for } i = 1, 2, 3 \quad (9.34)$$

and thus

$$\kappa_{D_{k+2(\ell)}, i} = \kappa_{D_{k(u)}, i} \quad \text{for } i = 1, 2, 3. \quad (9.35)$$

However, we note that for arbitrary  $q$ ,

$$P(D_{m_1, m_2}, q) \neq P(D_{m_2, m_1}, q) \quad \text{unless } m_1 = m_2. \quad (9.36)$$

## X. THE FAMILY $D_{m-4, 0}$

We proceed to examine a number of different  $D_{m_1, m_2}$  families of planar triangulations, with  $m_2$  held fixed. Then, we will analyze analogous families with  $m_1$  held fixed, and finally, we will investigate families in which both  $m_1$  and  $m_2$  vary together, and are related in a linear manner. For a given graph  $D_{m_1, m_2}$ , one can use either our general result for  $P(D_{m_1, m_2}, q)$  above or either of the one-parameter reductions, (9.22) or (9.29). However, we shall be interested in the limits  $m_1 \rightarrow \infty$  with  $m_2$  fixed, and  $m_2 \rightarrow \infty$  with  $m_1$  fixed, and, to study these, it is convenient to use the one-parameter reductions of our general formula.

TABLE III: Values of the ratios  $r(D_{m_1,\infty})$  and  $r(D_{\infty,m_2})$ . Note that  $r(D_{k,\infty}) = r(D_{\infty,k-2})$  for  $k \geq 2$ .

$r(D_{\infty,m_2}), r(D_{m_1,\infty})$	analytic	numerical
$r(D_{\infty,0})$	$-4 + 2\sqrt{5}$	0.472136
$r(D_{\infty,1})$	$(-15 + 7\sqrt{5})/2$	0.326238
$r(D_{\infty,2}) = r(D_{0,\infty})$	$-13 + 6\sqrt{5}$	0.416408
$r(D_{\infty,3}) = r(D_{1,\infty})$	$-22 + 10\sqrt{5}$	0.360680
$r(D_{\infty,4}) = r(D_{2,\infty})$	$(-73 + 33\sqrt{5})/2$	0.395122
$r(D_{\infty,5}) = r(D_{3,\infty})$	$-60 + 27\sqrt{5}$	0.373835
$r(D_{\infty,6}) = r(D_{4,\infty})$	$-98 + 44\sqrt{5}$	0.386991
$r(D_{\infty,7}) = r(D_{5,\infty})$	$(-319 + 143\sqrt{5})/2$	0.378860
$r(D_{\infty,8}) = r(D_{6,\infty})$	$-259 + 116\sqrt{5}$	0.383885
$r(D_{\infty,9}) = r(D_{7,\infty})$	$-420 + 188\sqrt{5}$	0.380780
$r(D_{\infty,10}) = r(D_{8,\infty})$	$(-1361 + 609\sqrt{5})/2$	0.382700
$r(D_{\infty,\infty})$	$(3 - \sqrt{5})/2$	0.381966

We begin with a study of the family  $D_{m_1,0} \equiv D_{m-4,0}$  with  $m_1 \geq 0$ , i.e.,  $m \geq 4$ . From (9.1), we have  $n(D_{m-4,0}) = m + 5$ . For the coefficients that enter into the equation (3.16), our general formulas (9.23)-(9.26) yield

$$\kappa_{D_{0(\ell)},1} = \kappa_{D_{0(\ell)},2} = \kappa_{D_{0(\ell)},3} = q^3 - 9q^2 + 29q - 32 \quad (10.1)$$

(equal to  $\lambda_{TC}$ ). Because these coefficients  $\kappa_{D_{0(\ell)},j}$  are all the same,  $\lambda_{TC}$  is a common factor, so for all  $m$ , the three zeros of  $\lambda_{TC}$  are zeros of  $P(D_{m-4,0}, q)$ . Of these, one is real, namely  $q_w$ , given in (3.25). In accordance with our general analysis above,  $P(D_{m-4,0}, q)$  also has a zero, denoted  $q_z$ , that approaches  $\tau + 1$  as  $m$  increases. We list this zero for  $m = 1$  to  $m = 16$  in Table IV. As is evident from this table, for odd (even)  $m$ , this zero is slightly above (below)  $\tau + 1$ .

For the evaluation at  $q = \tau + 1$ , we have

$$P(D_{m-4,0}, \tau + 1) = (-4 + 2\sqrt{5})(\tau - 1)^m + (3 - \sqrt{5})(\tau - 2)^m, \quad (10.2)$$



TABLE IV: Location of zero  $q_z$  of  $P(D_{m-4,0}, q)$  closest to  $\tau + 1$ , as a function of the number of vertices,  $n = m + 5$ .

$n$	$q_z$	$q_z - (\tau + 1)$
10	2.677815	0.05978
11	2.594829	-0.02321
12	2.636118	0.01808
13	2.609130	-0.8904e-2
14	2.624356	0.6322e-2
15	2.614541	-0.3493e-2
16	2.620356	2.322e-3
17	2.616673	-1.361e-3
18	2.618905	0.8713e-3
19	2.617509	-0.5254e-3
20	2.618364	0.3301e-3
21	2.617832	-2.017e-4
22	2.618160	1.2560e-4
23	2.617957	-0.7725e-4
24	2.618082	0.4790e-4
25	2.618004	-2.954e-5

so that

$$r(D_{m-4,0}) = -4 + 2\sqrt{5} + (3 - \sqrt{5}) \left( \frac{1 - \sqrt{5}}{2} \right)^m. \quad (10.3)$$

Hence,

$$r(D_{\infty,0}) = -4 + 2\sqrt{5} = 0.4721359... \quad (10.4)$$

with  $a_{D_0(\ell)} = 1$ .

## XI. THE FAMILY $D_{m-4,1}$

We continue with a study of the family  $D_{m,1} \equiv D_{m-4,1}$ . From (9.1), we have  $n(D_{m-4,1}) = m + 6$ . Our general formulas (9.23)-(9.26) give the coefficients  $\kappa_{D_{1(\ell)},j}$  as

$$\kappa_{D_{1(\ell)},1} = (q-3)(q^3 - 9q^2 + 30q - 35) , \quad (11.1)$$

$$\kappa_{D_{1(\ell)},2} = q^4 - 12q^3 + 58q^2 - 133q + 119 , \quad (11.2)$$

and

$$\kappa_{D_{1(\ell)},3} = -(q-3)(2q^2 - 14q + 25) . \quad (11.3)$$

If  $m$  is even, then  $P(D_{m-4,1}, q)$  has not only the factor  $q(q-1)(q-2)(q-3)$ , but also an additional factor of  $(q-3)$ . According to our general analysis above,  $P(D_{m-4,1}, q)$  has a real zero that approaches  $\tau + 1$  as  $m \rightarrow \infty$ . We also derived the result that for sufficiently large  $m$ , a chromatic polynomial of the form (3.16) has another real zero in the interval  $[q_w, 3)$  if and only if  $\kappa_{G_{pt},3}$  has a zero in this interval. For the present family,  $\kappa_{D_{1(\ell)},3}$  has zeros at  $q = 3$  and the complex-conjugate pair  $q = (7 \pm i)/2$ , but does not have a zero in the interval  $[q_w, 3)$ , in accordance with the fact that  $P(D_{m-4,1}, q)$  also does not have a zero in this interval.

For the evaluation at  $\tau + 1$ , we compute

$$P(D_{m-4,1}, \tau + 1) = \left( \frac{25 + 11\sqrt{5}}{2} \right) (\tau - 1)^m + (-9 + 4\sqrt{5}) (\tau - 2)^m , \quad (11.4)$$

so that

$$r(D_{m-4,1}) = \frac{-15 + 7\sqrt{5}}{2} + \left( \frac{11 - 5\sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^m . \quad (11.5)$$

and

$$r(D_{\infty,1}) = \frac{-15 + 7\sqrt{5}}{2} = 0.3226238 \quad (11.6)$$

with  $a_{D_{1(\ell)}} = 1$ .

## XII. THE FAMILY $D_{m-4,2}$

We next study the family  $D_{m,2} \equiv D_{m-4,2}$  with  $m_1 \geq 0$ , i.e.,  $m \geq 4$ . Note that  $D_{1,2}$  is the same as the graph denoted  $G_{ce12}$  in Fig. 8 of [1]. From (9.1), we have  $n(D_{m-4,2}) = m + 7$ . For this family our general results give

$$\kappa_{D_{2(\ell)},1} = q^5 - 15q^4 + 94q^3 - 303q^2 + 498q - 332 , \quad (12.1)$$

$$\kappa_{D_{2(\ell)},2} = q^5 - 15q^4 + 95q^3 - 317q^2 + 553q - 398 , \quad (12.2)$$

and

$$\kappa_{D_{2(\ell)},3} = -(q^4 - 16q^3 + 91q^2 - 225q + 206) . \quad (12.3)$$

In Table V we list (real) zeros of  $P(D_{m-4,2}, q)$  in the interval  $q \in [q_w, 3)$  as a function of  $n$ . As proved above, one zero approaches  $\tau + 1$  as  $m \rightarrow \infty$ . In the same limit, our general analysis above shows that  $P(D_{m-4,2}, q)$  has real zero(s) in the interval  $[q_w, 3)$  corresponding to the zeros of  $\kappa_{D_{2(\ell)},3}$  in this interval. This quartic polynomial has a zero at

$$q = 2.7227000945... \quad (12.4)$$

together with one more real zero at  $q = 6.955106...$ , outside the interval  $[q_w, 3)$ , and a complex-conjugate pair. Hence,  $P(D_{m-4,2}, q)$  has another zero in the interval  $[q_w, 3)$ , which is present for  $m \geq 5$ , and this approaches the zero of  $\kappa_{D_{2(\ell)},3}$  given in (12.4) as  $m \rightarrow \infty$ . For even  $m \geq 6$ , i.e., odd  $n \geq 13$ , the real zero near to this asymptotic value (12.4) increases toward it, while for odd  $m \geq 5$ , i.e., even  $n \geq 12$ , the nearby real decreases toward the asymptotic value. As noted above, the graph  $D_{0,2}$  coincides with the graph  $G_{CM,1}$  of [1], for which there is no real zero close to  $\tau + 1$ ; instead, the zeros closest to  $\tau + 1$  comprise a complex-conjugate pair at  $q = 2.641998 \pm 0.014795i$ . Correspondingly, for  $D_{0,2}$  there is no zero  $q'_z$  in the interval  $[q_w, 3)$ . For all of the  $D_{m-4,2}$  with  $m \geq 5$  in Table V, the first real zero  $q_z$  in the interval  $[q_w, 3)$  is, in fact, the closest to  $\tau + 1$ .

The evaluation at  $q = \tau + 1$  yields

$$P(D_{m-4,2}, \tau + 1) = \left( \frac{-69 + 31\sqrt{5}}{2} \right) (\tau - 1)^m + (27 - 12\sqrt{5}) (\tau - 2)^m , \quad (12.5)$$

so that

$$r(D_{m-4,2}) = -13 + 6\sqrt{5} + \left( \frac{21 - 9\sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^m . \quad (12.6)$$

Hence,

$$r(D_{\infty,2}) = -13 + 6\sqrt{5} = 0.41640786... \quad (12.7)$$

with  $a_{D_{2(\ell)}} = 1$ .

### XIII. THE FAMILY $D_{m-4,3}$

The final family that we study in this series is  $D_{m_1,3} \equiv D_{m-4,3}$ , with  $m_1 \geq 0$ , i.e.,  $m \geq 4$ . From (9.1), it follows that the graph  $D_{m-4,3}$  has  $n(D_{m-4,3}) = m + 8$ . The graphs  $D_{1,3}$  and  $D_{2,3}$  are shown in Figs. 7 and 8.

For the coefficients  $\kappa_{D_{3(\ell)},j}$  we have

$$\kappa_{D_{3(\ell)},1} = (q - 3)(q^2 - 5q + 7)(q^3 - 10q^2 + 38q - 49) , \quad (13.1)$$

TABLE V: Location of real zeros of  $P(D_{m-4,2}, q)$  in the interval  $q \in [q_w, 3)$ , as a function of the number of vertices,  $n = m + 7$ . Here the notation nz means that there is no second real zero in the interval  $[q_w, 3)$ .

$n$	$q_z$	$q'_z$
11	c.c. pair	nz
12	2.614614	2.818897
13	2.621801	2.689610
14	2.616506	2.762806
15	2.619226	2.705035
16	2.6174035	2.741044
17	2.618462	2.713055
18	2.617785	2.731543
19	2.618194	2.717464
20	2.617938	2.727100
21	2.618094	2.719886
22	2.617997	2.724931
23	2.618057	2.721202
24	2.618020	2.723845

$$\kappa_{D_{3(\ell)},2} = q^6 - 18q^5 + 141q^4 - 613q^3 + 1551q^2 - 2152q + 1271 , \quad (13.2)$$

and

$$\kappa_{D_{3(\ell)},3} = -(q-3)^2(q^3 - 12q^2 + 48q - 67) . \quad (13.3)$$

In Table VI we list real zeros of  $P(D_{m-4,3}, q)$  in the interval  $q \in [q_w, 3)$ . As is evident in this table, in addition to the zero that approaches  $\tau + 1$ , there is a second real zero in this interval if and only if  $m$  (and hence  $n$ ) is even. We can prove that, for the subset of  $D_{m-4,3}$  with even  $m$  where this second zero is present, it approaches  $q = 3$  from below as  $m \rightarrow \infty$ . The proof is as follows. According to our little theorem above on real chromatic zeros besides the one (or complex pair) that approach  $\tau + 1$  as  $m \rightarrow \infty$ , there is a second zero in the interval  $q \in [q_w, 3)$  if and only if  $\kappa_{G_{pt},3}$  has a real zero in this interval. Now for the  $D_{m-4,3}$  family,  $\kappa_{D_{3(\ell)},3}$  has no real zero in the interval  $[q_w, 3)$ . (Its zeros are at  $q = 3$ , with

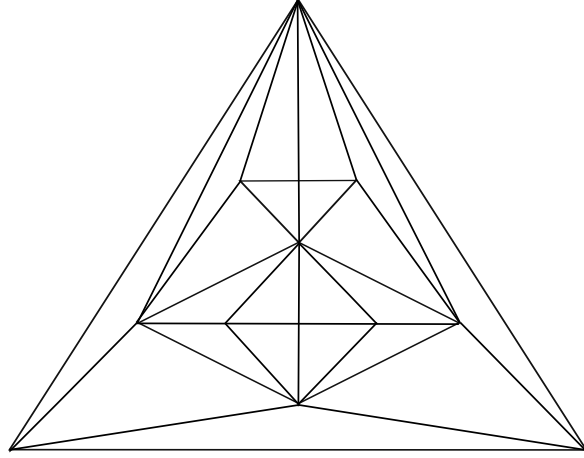


FIG. 7: Graph  $D_{1,3}$ .

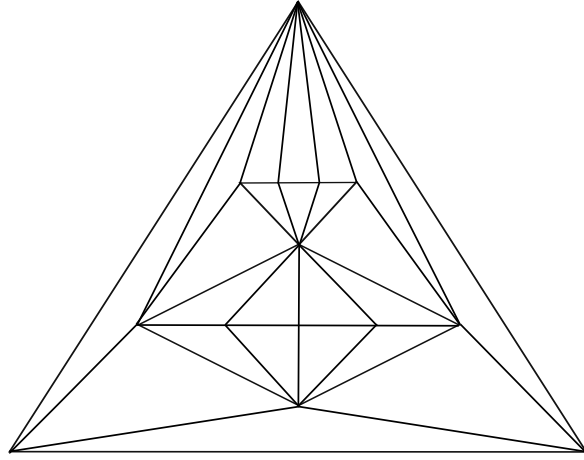


FIG. 8: Graph  $D_{2,3}$ .

multiplicity 2, at  $q = 5.44224957\dots$  and at  $q \simeq 3.278875 \pm 1.249025$ .) Hence, according to our little theorem, as  $m \rightarrow \infty$  on even integers, in addition to the real zero that is near to  $\tau + 1$ , the other zero must approach 3, so that in this limit, there is no other zero in the interval  $[q_w, 3)$ . This family may thus be added to the two known families given (in his Theorem 4) by Woodall in Ref. [17] as examples of one-parameter families of graphs, each of which has a chromatic zero that approaches 3 from below as the parameter ( $m$  here) goes to infinity.

The evaluation at  $q = \tau + 1$  yields

$$P(D_{m-4,3}, \tau + 1) = (94 - 42\sqrt{5})(\tau - 1)^m + (-76 + 34\sqrt{5})(\tau - 2)^m, \quad (13.4)$$

TABLE VI: Location of real zeros of  $P(D_{m-4,3}, q)$  in the interval  $q \in [q_w, 3)$ , as a function of the number of vertices,  $n = m + 8$ . Notation nz means that there is no second zero in this interval.

$n$	$q_z$	$q'_z$
12	2.614614	2.818897
13	2.619530	nz
14	2.616973	2.847527
15	2.618625	nz
16	2.617649	2.866268
17	2.618264	nz
18	2.617889	2.880165
19	2.618122	nz
20	2.617979	2.890985
21	2.618068	nz
22	2.618013	2.899700
23	2.618057	nz
24	2.618020	2.906905
25	2.618039	nz
26	2.618031	2.912980

so that

$$r(D_{m-4,3}) = -22 + 10\sqrt{5} + (18 - 8\sqrt{5}) \left( \frac{1 - \sqrt{5}}{2} \right)^m. \quad (13.5)$$

Hence,

$$r(D_{\infty,3}) = -22 + 10\sqrt{5} = 0.36067977... \quad (13.6)$$

with  $a_{D_{3(\ell)}} = 1$ .

#### XIV. THE FAMILY $D_{0,m-2}$

As an example of families with  $m_1$  fixed and variable  $m_2$  we discuss the family  $D_{0,m_2} \equiv D_{0,m-2}$ . A graph in this family has  $n(D_{0,m-2}) = m + 7$ . In accord with our result (9.35),

$$\kappa_{D_{0(u)},i} = \kappa_{D_{2(u)},i} \quad \text{for } i = 1, 2, 3. \quad (14.1)$$

$P(D_{0,m-2}, q)$  has a real zero near to  $\tau + 1$ , which approaches this point as  $m \rightarrow \infty$ . Furthermore, since the coefficient  $\kappa_{D_{0(u)},3}$  has a real zero in the interval  $[q_w, 3)$ , at the value in (12.4), it follows from our general analysis above that for sufficiently large  $m$ ,  $P(D_{0,m-2}, q)$  has a real zero that approaches this value. These zeros approach their respective values in a manner similar to that discussed for the family  $D_{m-4,2}$ .

#### XV. A SYMMETRIC TWO-PARAMETER FAMILY $S_{m_1,m_2}$

In this section we study a two-parameter family of planar triangulations  $S_{m_1,m_2}$  which are symmetric under interchange of the parameters:

$$S_{m_1,m_2} = S_{m_2,m_1}. \quad (15.1)$$

We show the lowest member of this family,  $S_{0,0}$  in Fig. 9 and another member,  $S_{1,2} = S_{2,1}$  in Fig. 10. From these it is clear how to construct the general graph  $S_{m_1,m_2}$  in this family. We have  $n(S_{m_1,m_2}) = m_1 + m_2 + 7$ . The chromatic number is  $\chi(S_{m_1,m_2}) = 4$ , and  $P(S_{m_1,m_2}, q)$  contains the factor  $P(K_4, q) = q(q-1)(q-2)(q-3)$ .

We have calculated  $P(S_{m_1,m_2}, q)$  and find that it involves the same three  $\lambda$ 's as in (3.1), but with an interestingly different form than  $P(D_{m_1,m_2}, q)$ . Given the symmetry (15.1), it follows that the coefficients in  $P(S_{m_1,m_2}, q)$  satisfy

$$c_{S,ij} = c_{S,ji}. \quad (15.2)$$

Consequently, although there are nine terms of the form  $\lambda_i^{m_1} \lambda_j^{m_2}$  in  $P(S_{m_1,m_2}, q)$ , there are only six independent coefficients  $c_{S,ij}$  to begin with, and we find that two of these vanish, so that there are only four independent, nonvanishing coefficients  $c_{S,ij}$ . Explicitly, we calculate

$$c_{S,ij} = q \bar{c}_{S,ij}, \quad (15.3)$$

with

$$c_{S,11} = c_{S,13} = c_{S,31} = 0, \quad (15.4)$$

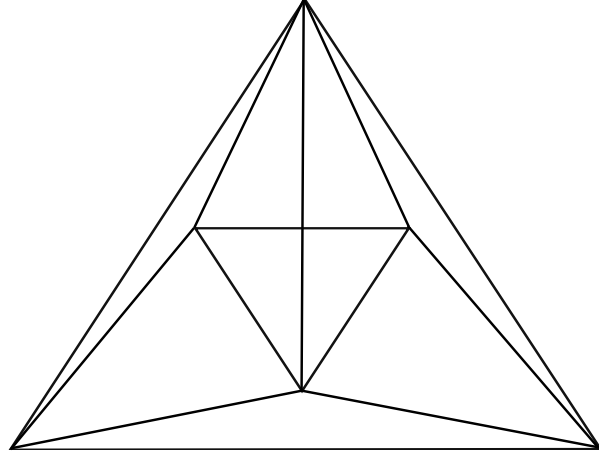


FIG. 9: Graph  $S_{0,0}$ .

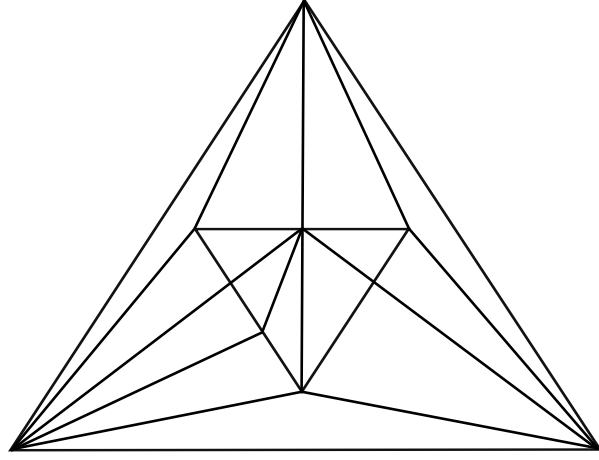


FIG. 10: Graph  $S_{2,1}$ .

$$\bar{c}_{S,22} = \frac{(q-1)(q-3)^6}{q-2} , \quad (15.5)$$

$$\bar{c}_{S,12} = \bar{c}_{S,21} = (q-2)^3(q-3)^2 , \quad (15.6)$$

$$\bar{c}_{S,23} = \bar{c}_{S,32} = \frac{(q-3)^2(q^2-3q+1)}{q-2} , \quad (15.7)$$

and

$$\bar{c}_{S,33} = \frac{(q-1)(q-3)(q^2-3q+1)}{q-2} . \quad (15.8)$$



so that

$$\begin{aligned}
P(S_{m_1, m_2}, q) &= c_{S,22} \lambda_2^{m_1+m_2} + c_{S,33} \lambda_3^{m_1+m_2} \\
&+ c_{S,12} (\lambda_1^{m_1} \lambda_2^{m_2} + \lambda_2^{m_1} \lambda_1^{m_2}) + c_{S,23} (\lambda_2^{m_1} \lambda_3^{m_2} + \lambda_3^{m_1} \lambda_2^{m_2})
\end{aligned} \tag{15.9}$$

As before, the poles cancel in the calculation of  $P(S_{m_1, m_2}, q)$  and, furthermore, these coefficients satisfy the requisite identities so that  $P(S_{m_1, m_2}, q) = 0$  for  $q = 1, 2, 3$ . These are special cases of (7.10) and (7.13) that incorporate the properties that  $c_{S,ij} = c_{S,ji}$  and  $c_{S,11} = c_{S,13} = c_{S,31} = 0$ , namely,

$$c_{S,22} = 0, \quad c_{S,33} = 0, \quad c_{S,12} + c_{S,23} = 0 \quad \text{at } q = 1 \tag{15.10}$$

and

$$c_{S,22} + 2c_{S,12} + c_{S,33} = 0, \quad c_{S,12} = 0 \quad \text{at } q = 2. \tag{15.11}$$

As a special case of (7.17) we also have

$$c_{S,23} = c_{S,32} = c_{S,33} = 0 \quad \text{at } q = \tau + 1. \tag{15.12}$$

Finally, the condition  $P(S_{m_1, m_2}, 3) = 0$  is equivalent to

$$c_{S,33} = 0 \quad \text{at } q = 3 \tag{15.13}$$

For the comparison of  $P(S_{m_1, m_2}, q)$  at  $q = \tau + 1$  with the Tutte upper bound  $(\tau - 1)^{m_1+m_2+2}$ , we have

$$\begin{aligned}
r(S_{m_1, m_2}) &= \left| (9 - 4\sqrt{5}) \left( \frac{1 - \sqrt{5}}{2} \right)^{m_1+m_2} \right. \\
&+ \left. (-2 + \sqrt{5}) \left[ \left( \frac{1 - \sqrt{5}}{2} \right)^{m_1} + \left( \frac{1 - \sqrt{5}}{2} \right)^{m_2} \right] \right|
\end{aligned} \tag{15.14}$$

This decreases (non-monotonically) in magnitude as  $m_1$  increases for fixed  $m_2$  and as  $m_2$  increases for fixed  $m_1$ , approaching zero exponentially rapidly as either of these parameters goes to infinity. Thus,

$$\lim_{m_1 \rightarrow \infty} r(S_{m_1, m_2}) = \lim_{m_2 \rightarrow \infty} r(S_{m_1, m_2}) = 0. \tag{15.15}$$

It is also of interest to analyze the one-parameter reductions of  $P(S_{m_1, m_2}, q)$  for variable  $m_1$  and fixed  $m_2$  and vice versa. These yield identical results, because of the symmetry (15.1). Hence, without loss of generality we consider variable  $m_1$  and fixed  $m_2$  and find that

in this case  $P(S_{m_1, m_2}, q)$  reduces to (3.16) with  $m = m_1 + 2$ , which we write as  $P(S_{m-2, m_2}, q)$ . Since the coefficients only depend on  $m_2$  and not  $m$ , we denote them by  $c_{S_{m_2}, i}$ . They are given by

$$c_{S_{m_2}, i} = \lambda_i^{-2} \sum_{j=1}^3 c_{S, ij} \lambda_j^{m_2} \quad (15.16)$$

Thus, in terms of the corresponding  $\kappa_{S_{m_2}, i}$ ,

$$\kappa_{S_0, 1} = (q-2)(q-3)^2, \quad (15.17)$$

$$\kappa_{S_0, 2} = \lambda_{TC} = q^3 - 9q^2 + 29q - 32, \quad (15.18)$$

$$\kappa_{S_0, 3} = 2(q-3). \quad (15.19)$$

and so forth for higher values of  $m_2$ .

This family exhibits a number of interesting properties. Among these is the fact that out of the possible  $3^2$  terms in (8.1) for  $p = 2$ , some may be absent because of vanishing coefficients  $c_{G, ij}$ . In particular, the term  $c_{G, 11} \lambda_1^{m_1+m_2}$  that would be dominant in the limit where the parameters  $m_1 \rightarrow \infty$  and  $m_2 \rightarrow \infty$ , may be absent, so that in this limit,  $r(G_{\infty, \infty})$  may be zero.

## XVI. FAMILIES OF THE FORM $G_{pt, m_1, m_2}$ WITH $m_1 = m_2$

In previous sections we have analyzed the chromatic polynomials of special cases of two-parameter families of planar triangulations  $G_{pt, m_1, m_2}$  as a function of  $m_1$  with  $m_2$  held fixed, and vice versa and shown how they reduce to (2.9) with  $j_{max} = 3$ . A different type of special case in which  $G_{pt, m_1, m_2}$  reduces to a one-parameter family is obtained by requiring that  $m_1$  and  $m_2$  be linearly related to each other. The simplest such example of this type of reduction is the diagonal case obtained by requiring that  $m_1 = m_2$ . For general families  $G_{pt, m_1, m_2}$  that satisfy (7.15) and for which  $P(G_{pt, m_1, m_2}, q)$  is of the form (7.1), it follows that  $n(G_{pt, k, k}) = 2k + \beta$  and that  $P(G_{pt, m_1, m_1}, q)$  reduces, to the form (2.9) with  $j_{max} = 6$ . We use the shorthand  $G_d$  to denote a generic  $G_{pt, m_1, m_1}$ . We have

$$P(G_{d, m_1, m_1}, q) = \sum_{j=1}^6 c_{G_d, j} (\lambda_{G_d, j})^m \quad (16.1)$$

where  $m = m_1 + \delta m$ , with  $\delta m$  depending on the family, and

$$\lambda_{G_d, 1} = \lambda_1^2 = (q-2)^2, \quad (16.2)$$

$$\lambda_{G_d,2} = \lambda_2^2 = (q-3)^2 , \quad (16.3)$$

$$\lambda_{G_d,3} = \lambda_3^2 = 1 , \quad (16.4)$$

$$\lambda_{G_d,4} = \lambda_1 \lambda_2 = (q-2)(q-3) , \quad (16.5)$$

$$\lambda_{G_d,5} = \lambda_1 \lambda_3 = -(q-2) , \quad (16.6)$$

and

$$\lambda_{G_d,6} = \lambda_2 \lambda_3 = -(q-3) . \quad (16.7)$$

The corresponding coefficients are

$$c_{G_d,j} = q\bar{c}_{G_d,j} \quad \text{for } j = 1, \dots, 6, \quad (16.8)$$

with

$$c_{G_d,1} = c_{G,11} , \quad (16.9)$$

$$c_{G_d,2} = c_{G,22} , \quad (16.10)$$

$$c_{G_d,3} = c_{G,33} , \quad (16.11)$$

$$c_{G_d,4} = c_{G,12} + c_{G,21} , \quad (16.12)$$

$$c_{G_d,5} = c_{G,13} + c_{G,31} , \quad (16.13)$$

and

$$c_{G_d,6} = c_{G,23} + c_{G,32} . \quad (16.14)$$

The coefficients  $c_{G_d,i}$ ,  $i = 1, \dots, 6$  satisfy various conditions that follow from those that we have derived for the coefficients  $c_{G,ij}$  in (7.7), (7.10), (7.13), and (7.17). These are

$$c_{G_d,1} + c_{G_d,3} + c_{G_d,5} = 0, \quad c_{G_d,2} = 0,$$

$$c_{G_d,4} + c_{G_d,6} = 0 \quad \text{at } q = 1 , \quad (16.15)$$

$$c_{G_d,1} = 0, \quad c_{G_d,2} + c_{G_d,3} + c_{G_d,6} = 0,$$

$$c_{G_d,4} + c_{G_d,6} = 0 \quad \text{at } q = 2 , \quad (16.16)$$

and

$$c_{G_d,3} = c_{G_d,5} = c_{G_d,6} = 0 \quad \text{at } q = \tau + 1 . \quad (16.17)$$

Hence,

$$c_{G_d,2} \quad \text{contains the factor } q-1 , \quad (16.18)$$

$$c_{G_d,1} \quad \text{contains the factor } q-2 , \quad (16.19)$$

and

$$c_{G_d,i} \quad \text{contains the factor } q^2 - 3q + 1 \quad \text{if } i = 3, 5, 6 . \quad (16.20)$$

## XVII. THE FAMILIES $D_{m_1, m_2}$ AND $S_{m_1, m_2}$ WITH $m_1 = m_2$

We now discuss two explicit examples of the diagonal special case of a two-parameter planar triangulation family, namely  $D_{m_1, m_2}$  and  $S_{m_1, m_2}$  with  $m_1 = m_2$ . We shall use the shorthand notation  $D_d$  and  $S_d$  to refer to these entire respective families. From (9.1), we have  $n(D_{m_1, m_1}) = 2m_1 + 9$  and  $n(S_{m_1, m_1}) = 2m_1 + 7$ .

The chromatic polynomial  $P(D_{m_1, m_1}, q)$  has the form (16.1) with  $j_{max} = 6$  and  $\delta m = 0$ , i.e.,  $m = m_1$ . The lowest member of this family, the graph  $D_{0,0}$ , was shown in Fig. 4 and  $D_{2,2}$  was shown in Fig. 6. The chromatic number  $\chi(D_{m,m})$  is 3 if  $m$  is even and 4 if  $m$  is odd. The coefficients are

$$\bar{c}_{D_d, i} = \bar{c}_{D, ii} \quad \text{for } i = 1, 2, 3, \quad (17.1)$$

$$\bar{c}_{D_d, 4} = \bar{c}_{12} + \bar{c}_{21} = (q-2)(q-3)^4(2q^2 - 10q + 13), \quad (17.2)$$

$$\bar{c}_{D_d, 5} = \bar{c}_{13} + \bar{c}_{31} = \frac{(q-2)(q^2 - 4q + 5)(q^2 - 3q + 1)}{q-1}, \quad (17.3)$$

and

$$\bar{c}_{D_d, 6} = \bar{c}_{23} + \bar{c}_{32} = -\frac{(q-3)^2(q-5)(q^2 - 6q + 10)(q^2 - 3q + 1)}{q-2}, \quad (17.4)$$

Since  $\chi(D_{m,m}) = 4$  for odd  $m$ , it follows that

$$c_{D_d, 1} - c_{D_d, 3} + c_{D_d, 5} = 0 \quad \text{for } q = 3. \quad (17.5)$$

We calculate

$$\begin{aligned} P(D_{m,m}, \tau + 1) &= (\tau + 1) \left[ \left( \frac{47 - 21\sqrt{5}}{2} \right) (\tau - 1)^{2m} \right. \\ &\quad \left. + \left( \frac{1165 - 521\sqrt{5}}{2} \right) (\tau - 2)^{2m} + (-360 + 161\sqrt{5}) [(\tau - 1)(\tau - 2)]^m \right] \end{aligned} \quad (17.6)$$

The ratio  $r(D_{m,m})$  is

$$r(D_{m,m}) = \frac{3 - \sqrt{5}}{2} + \left( \frac{65 - 29\sqrt{5}}{2} \right) \left( \frac{1 - \sqrt{5}}{2} \right)^{2m} + (-20 + 9\sqrt{5}) \left( \frac{1 - \sqrt{5}}{2} \right)^m. \quad (17.7)$$

Hence, defining  $r(D_d, \infty) = \lim_{m \rightarrow \infty} r(D_{m,m})$ , we have

$$r(D_d, \infty) = \frac{3 - \sqrt{5}}{2} = 0.381966... \quad (17.8)$$

and  $a_{D_d} = 1$ .

In contrast, the chromatic polynomial  $P(S_{m_1, m_1}, q)$  has the form (2.9) with  $j_{max} = 4$ . The lowest member of this family, the graph  $S_{0,0}$ , was shown in Fig. 9. The chromatic number  $\chi(S_{m,m}) = 4$ . The expression for  $P(S_{m,m}, q)$  follows immediately from (15.9) and has  $j_{max} = 4$ ,

$$P(S_{m,m}, q) = c_{S,22}(\lambda_2)^{2m} + c_{S,33} + 2c_{S,12}(\lambda_1\lambda_2)^m + 2c_{S,23}(\lambda_2\lambda_3)^m, \quad (17.9)$$

where we used the fact that  $(\lambda_3)^{2m} = 1$ . The ratio  $r(S_{m,m})$  follows from (15.14), with the result that  $r(S_{\infty,\infty}) = 0$ .

### XVIII. THE FAMILY $F_m$

In this section we construct and study a family of planar triangulations, denoted  $F_m$ , with the property that  $P(F_m, q)$  has the form (2.9) with  $j_{max} = 3$ , but the  $\lambda_{F,j}$  are not given by (3.1), but instead are roots of a certain cubic equation. The number of vertices is  $n(F_m) = m + 4$ . This family is useful as a contrast to the other one-parameter families of planar triangulations with chromatic polynomials of the form (2.9) that we have constructed. The construction of members of this family is somewhat more complicated than that of the other families analyzed in this paper, and accordingly, for illustration we include several graphs, namely  $F_m$  with  $m = 3, 4, 5$ , are shown in Figs. 11-13. As these show, starting from a given member  $F_m$ , one constructs the next higher member  $F_{m+1}$  in an interleaved manner, first adding a new set of edges one of which emanates from the lower left-hand vertex of the graph, and then a new set of edges one of which emanates from the uppermost vertex, and so forth. We note that in contrast to the previous planar triangulations with chromatic polynomials of the form (3.16), the degrees of the vertices remain bounded as  $m \rightarrow \infty$  for this family.

As in earlier works [8–10], it is most convenient to express the  $P(F_m, q)$  via a generating function,  $\Gamma(F, q, x)$ , which is a rational function in  $q$  and an auxiliary expansion variable  $x$ , of the form

$$\Gamma(F, q, x) = \frac{\mathcal{N}(F, q, x)}{\mathcal{D}(F, q, x)}, \quad (18.1)$$

where the numerator and denominator are

$$\mathcal{N}(F, q, x) = a_{F,0} + a_{F,1}x + a_{F,2}x^2 \quad (18.2)$$

and

$$\mathcal{D}(F, q, x) = 1 + b_{F,1}x + b_{F,2}x^2 + b_{F,3}x^3. \quad (18.3)$$

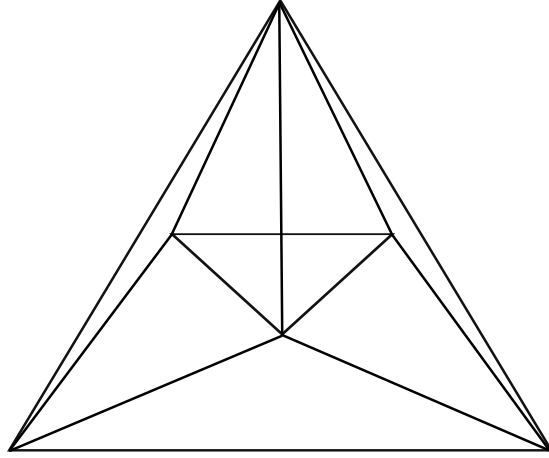


FIG. 11: Graph  $F_3$ .

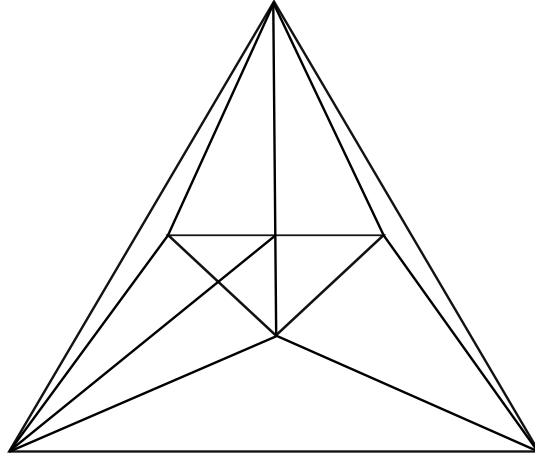


FIG. 12: Graph  $F_4$ .

with  $a_{F,j}$  and  $b_{F,j}$  being polynomials in  $q$ . The chromatic polynomial  $P(F_m, q)$  is then given as the coefficient in the Taylor series expansion of this generating function:

$$\Gamma(F, q, x) = \sum_{m=0}^{\infty} P(F_{m+1}, q) x^m \quad (18.4)$$

Using an iterative deletion-contraction method, we have determined this generating function. We find

$$a_{F,0} = q(q-1)(q-2)(q-3)^2 \quad (18.5)$$

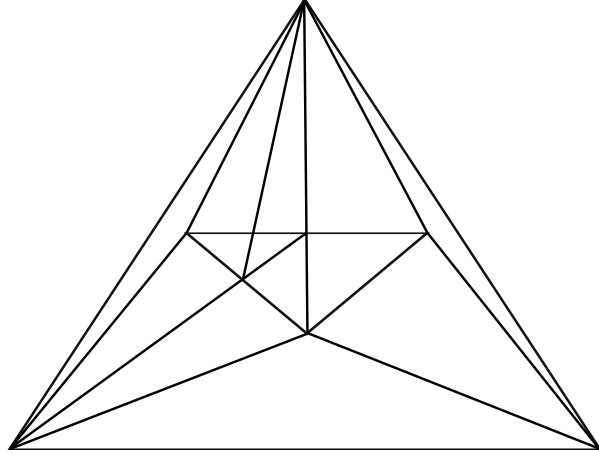


FIG. 13: Graph  $F_5$ .

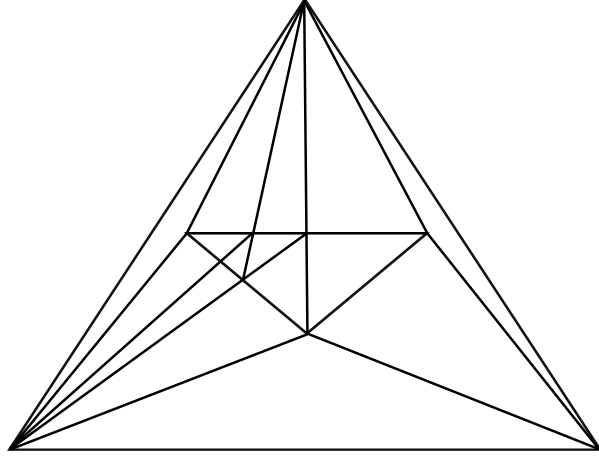


FIG. 14: Graph  $F_6$ .

$$a_{F,1} = q(q-1)(q-2)(2q-5) \quad (18.6)$$

$$a_{F,2} = q(q-1)(q-2)^2(q-3)^2 \quad (18.7)$$

$$b_{F,1} = -(q-3) \quad (18.8)$$

$$b_{F,2} = q-3 \quad (18.9)$$

and

$$b_{F,3} = -(q-2)(q-3) . \quad (18.10)$$

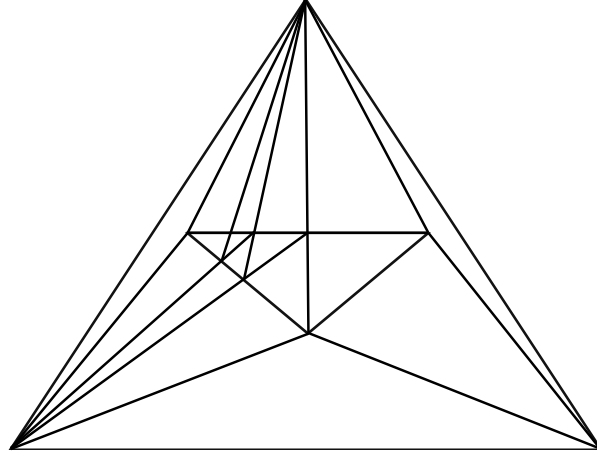


FIG. 15: Graph  $F_7$ .

The chromatic polynomial may also be expressed in the form of Eq. (2.9), with  $j_{max} = 3$ , namely

$$P(F_m, q) = \sum_{j=1}^3 c_{F,j} (\lambda_{F,j})^m . \quad (18.11)$$

Using Eq (2.13) (or (2.15)) of Ref. [9], one can calculate the coefficients  $c_{F,j}$  for  $j = 1, 2, 3$  from the generating function. Specifically, we have

$$c_{F,1} = \frac{(a_{F,0}\lambda_{F,1}^2 + a_{F,1}\lambda_{F,1} + a_{F,2})}{(\lambda_{F,1} - \lambda_{F,2})(\lambda_{F,1} - \lambda_{F,3})} \quad (18.12)$$

$$c_{F,2} = \frac{(a_{F,0}\lambda_{F,2}^2 + a_{F,1}\lambda_{F,2} + a_{F,2})}{(\lambda_{F,2} - \lambda_{F,1})(\lambda_{F,2} - \lambda_{F,3})} \quad (18.13)$$

and

$$c_{F,3} = \frac{(a_{F,0}\lambda_{F,3}^2 + a_{F,1}\lambda_{F,3} + a_{F,2})}{(\lambda_{F,3} - \lambda_{F,1})(\lambda_{F,3} - \lambda_{F,2})} . \quad (18.14)$$

As discussed before [8–10], the  $\lambda_{F,j}$ s appear via the factorized form of the denominator of the generating function,

$$\mathcal{D}(F, q, x) = \prod_{j=1}^3 (1 - \lambda_{F,j}x) . \quad (18.15)$$

Equivalently, the  $\lambda_{F,j}$ s are determined from the equation  $\xi^3 + b_{F,1}\xi^2 + b_{F,2}\xi + b_{F,3} = 0$ , i.e.,

$$\xi^3 + (3 - q)\xi^2 + (q - 3)\xi - (q - 2)(q - 3) = 0 . \quad (18.16)$$



Let us define

$$R_F = 3(4q^3 - 24q^2 + 76q - 93) \quad (18.17)$$

and

$$S_F = \left[ 4(q-3) \left( 2q^2 + 6q - 9 + 3\sqrt{R_F} \right) \right]^{1/3}. \quad (18.18)$$

With appropriate choices of branch cuts for the various fractional powers in (18.18), we have

$$\lambda_{F,1} = \frac{S_F}{6} + \frac{2(q-3)(q-6)}{3S_F} + \frac{q-3}{3}. \quad (18.19)$$

The other  $\lambda_{F,j}$ ,  $j = 2, 3$  can be written explicitly in a similar manner. Thus, this family is valuable as an illustration of a family of planar triangulation graphs with a chromatic polynomial of the form (2.9) and with  $\lambda$  terms that are different from those in (3.1) and, indeed, are nonpolynomial, in contrast to the families with chromatic polynomials of the form (2.8) or (3.16).

With regard to the evaluation of  $P(F_m, q)$  at  $q = \tau + 1$  (with an appropriate choice of branch cuts for the square and cube roots),  $\lambda_{F,1}$  and one of the other two roots of (18.16) comprise the complex-conjugate pair

$$\frac{1}{4} \left[ -1 + \sqrt{5} \pm (-38 + 18\sqrt{5})^{1/2} i \right] \quad (18.20)$$

with magnitude 0.485867..., while the third root of (18.16) is equal to  $-1$ . Since 0.485867.. is less than  $\tau - 1 = 0.6180...$ , the corresponding two coefficients do not have to, and do not, vanish at  $q = \tau + 1$ . Since the third root has magnitude greater than  $\tau - 1$ , its coefficient must vanish at  $q = \tau + 1$  in order for  $|P(F_m, \tau + 1)|$  to obey the Tutte upper bound (1.1). With these values of the  $\lambda_{F,j}$ 's at  $q = \tau + 1$ , the ratio  $r(F_m)$  vanishes (exponentially rapidly) as  $m \rightarrow \infty$  and  $r(F_\infty) = 0$ . This illustrates the general property that if  $G_{pt,m}$  is a family of planar triangulations with  $P(G_{pt,m}, q)$  of the form (2.9) and  $\alpha = 1$  in (2.1), and if none of the  $\lambda_{G_{pt,j}}$  has magnitude equal to  $\tau - 1$  when evaluated at  $q = \tau + 1$ , then, since (i) the  $\lambda_{G_{pt,j}}$  with  $|\lambda_{G_{pt,j}}| > \tau - 1$  have coefficients that must vanish, and (ii) the  $\lambda_{G_{pt,j}}$  with  $|\lambda_{G_{pt,j}}| < \tau - 1$  give zero contribution in the limit  $m \rightarrow \infty$ , it follows that  $r(G_{pt}) = 0$ . We calculate

$$a_F = 0.786151.. \quad (18.21)$$

The term  $\lambda_{F,1}$  is real and positive and is dominant for  $q > \tau + 2 = 3.618...$ . In this interval, the other two roots,  $\lambda_{F,j}$ ,  $j = 2, 3$  are complex, with smaller magnitudes. At  $q = \tau + 2$ ,  $\lambda_{F,1} = -1$  and  $|\lambda_{F,2}| = |\lambda_{F,3}| = 1$ , so all  $\lambda_{F,j}$  are degenerate in magnitude. Hence, in the notation of [8],  $q_c = \tau + 2$  for this family. At  $q = 3$ , all  $\lambda_{F,j} = 0$ ,  $j = 1, 2, 3$ , as is obvious from Eq. (18.16).

We exhibit the first few  $P(F_m, q)$ . For  $m = 1$ ,  $P(F_1, q) = a_{F,0}$ , as given above in (18.5). For  $m = 2$  to  $m = 6$ ,

$$P(F_2, q) = q(q-1)(q-2)(q^3 - 9q^2 + 29q - 32) \quad (18.22)$$

$$P(F_3, q) = q(q-1)(q-2)(q-3)(q^3 - 9q^2 + 30q - 35) \quad (18.23)$$

$$P(F_4, q) = q(q-1)(q-2)(q-3)(q^4 - 12q^3 + 58q^2 - 133q + 119) \quad (18.24)$$

$$P(F_5, q) = q(q-1)(q-2)(q-3)(q^5 - 15q^4 + 95q^3 - 317q^2 + 553q - 398) \quad (18.25)$$

$$P(F_6, q) = q(q-1)(q-2)(q-3)^2(q^5 - 15q^4 + 96q^3 - 327q^2 + 591q - 447) \quad (18.26)$$

As  $m$  increases further,  $P(F_m, q)$  has increasingly high powers of the factor  $(q-3)$ .

As with the other planar triangulation families, the  $F_m$  family has chromatic zeros near to  $\tau + 1$ . We find that these approach  $\tau + 1$  as  $m$  gets large. Depending on the value of  $m$ ,  $P(F_m, q)$  also may have real zeros in the interval  $[q_w, 3)$ . The complex zeros of  $P(F_m, q)$  form a complex-conjugate arc, with arc endpoints at the complex zeros of  $R_F$ , namely  $q$ ,  $q^* \simeq 1.9111 \pm 2.6502i$ .

## XIX. SOME IMPLICATIONS FOR STATISTICAL PHYSICS

One of the interesting aspects of the present work is its implications for nonzero ground state entropy of the Potts antiferromagnet. (For background on the Potts model, see Refs. [35, 36], [8].) This stems from the identity noted above,  $P(G, q) = Z(G, q, T = 0)_{PAF} = W_{tot}(G, q)$ . As above, we denote the formal limit of a family of graphs  $G$  as  $n(G) \rightarrow \infty$  by the symbol  $\{G\}$ . We recall that the entropy per vertex is given by  $S_0 = k_B \ln W$ , where  $W$  is the degeneracy per vertex, related to the total degeneracy of spin configurations of the zero-temperature Potts antiferromagnet (or equivalently the number of proper  $q$ -colorings of the graph) by  $W_{tot}$  by  $W(\{G\}, q) = \lim_{n \rightarrow \infty} [W_{tot}(G, q)]^{1/n}$ . We refer the reader to Ref. [8] for a discussion of a subtlety in this definition resulting from a certain noncommutativity that occurs for a special set of values of  $q$ , denoted as  $\{q_s\}$ , namely

$$\lim_{q \rightarrow q_s} \lim_{n \rightarrow \infty} [P(G, q)]^{1/n} \neq \lim_{n \rightarrow \infty} \lim_{q \rightarrow q_s} [P(G, q)]^{1/n} . \quad (19.1)$$

For the one-parameter families of planar triangulations considered here, this set of special values  $\{q_s\}$  includes  $q = 0, 1, 2, \tau + 1$  and, for cases where the chromatic number is 4, also  $q = 3$ . Because of this noncommutativity, it is necessary to specify the order of limits taken in defining  $W$ . For a particular value  $q = q_s$ , we thus define

$$W_{qn}(\{G\}, q_s) = \lim_{q \rightarrow q_s} \lim_{n \rightarrow \infty} [P(G, q)]^{1/n} \quad (19.2)$$

and

$$W_{nq}(\{G\}, q_s) = \lim_{n \rightarrow \infty} \lim_{q \rightarrow q_s} [P(G, q)]^{1/n} . \quad (19.3)$$

For real  $q \geq \chi(G_{pt,m})$ , both of these definitions are equivalent, and in this case we shall write  $W_{qn}(\{G\}, q_s) = W_{nq}(\{G\}, q_s) \equiv W(\{G\}, q_s)$ .

We generalize our calculations in [1] as follows. First, for a family of planar triangulations  $G_{pt,m}$  with  $P(G_{pt,m}, q)$  having the form (2.9) with  $j_{max} = 1$ ,  $W = [\lambda_{G_{pt}}]^{1/\alpha}$ . As an example, consider the family

$$R_m = P_m + P_2 , \quad (19.4)$$

with  $n = m + 2$  considered in [1]. Here  $P_m$  is the path graph with  $m$  vertices and the join  $G + H$  of two graphs  $G = (V_G, E_G)$  and  $H = (V_H, E_H)$  is defined as the graph with vertex set  $V_{G+H} = V_G \cup V_H$  and edge set  $E_{G+H}$  comprised of the union of  $E_G \cup E_H$  with the set of edges obtained by connecting each vertex of  $G$  with each vertex of  $H$ . Thus,  $R_1 = K_3$ ,  $R_2 = K_4$ , etc. The  $W$  function for the  $m \rightarrow \infty$  limit of this family is  $W(\{R\}, q) = q - 3$ , and  $S_0 > 0$  for  $q > 4$ . For the family  $TC_m$

$$W(\{TC\}, q) = (\lambda_{TC})^{1/3} . \quad (19.5)$$

where  $\lambda_{TC}$  was given in (3.24). The function  $\lambda_{TC}$  is a monotonically increasing function of  $q$ , which passes through zero at  $q = 2.54660\dots$  and increases through unity at  $q = 3$  so that (for the  $m \rightarrow \infty$  limit of this family)  $S_0 > 0$  for  $q > 3$ . For the family  $I_m$  of iterated icosahedra,

$$W(\{I\}, q) = (\lambda_I)^{1/9} , \quad (19.6)$$

where

$$\begin{aligned} \lambda_I = & (q - 3)(q^8 - 24q^7 + 260q^6 - 1670q^5 + 6999q^4 - 19698q^3 \\ & + 36408q^2 - 40240q + 20170) . \end{aligned} \quad (19.7)$$

The function  $\lambda_I$  vanishes at three real values of  $q$ , namely  $q = 2.618197\dots$  (i.e., slightly above  $\tau + 1$ ), at  $q = 3$ , and at  $q = 3.222458\dots$ . This function is positive for  $q > 3.222458\dots$  (as well

as in an interval  $2.618.. < q < 3$ ) and increases through unity as  $q$  increases through the value  $3.5133658..$ , so that in this latter interval,  $S_0 > 0$ .

A second general result is that for a family of planar triangulations  $G_{pt,m}$  with  $P(G_{pt,m}, q)$  having the form (3.16), it follows that (i)  $W_{qn}(\{G_{pt}\}, q) = q - 2$  for  $q > 3$ ; (ii) even in the presence of noncommutativity,  $W(\{G_{pt}\}, q) = q - 2$  for  $q \geq 4$ , so that  $S_0 > 0$  in this interval (and also in the interval  $q > 3$  if one uses  $W_{qn}(\{G_{pt}\}, q)$ ). This result applies, in particular, to the families  $B_m$ ,  $H_m$ ,  $L_m$ ,  $D_{m-4,2}$ , and  $D_{m-4,3}$ . Although the family  $P(D_{m,m}, q)$  is of the form (2.9) with  $j_{max} = 6$ , the dominant term for  $q > \chi(D_{m,m})$  is again  $q - 2$ , so that in this interval  $W(\{D_d\}, q) = q - 2$  for this family also. In contrast,  $P(S_{m_1,m_2}, q)$  has  $c_{S,11} = 0$  and hence lacks the term that would normally be dominant as  $m_1$  or  $m_2$  goes to infinity. In this case, for  $q \geq 4$  where there is no noncommutativity in limits, we find  $W(\{S\}, q) = \sqrt{(q-2)(q-3)}$ .

For the family  $F_m$ , we find that  $\lambda_1$  in Eq. (18.19) is dominant for  $q > q_c = \tau + 2 = 3.618...$ , so that in this region,

$$W(\{F\}, q) = \lambda_{F,1} \quad (19.8)$$

Furthermore, since  $\lambda_{F,1} > 1$  for real  $q > \tau + 2$ , it follows that  $S_0 > 0$  for (the  $m \rightarrow \infty$  limit of this family of graphs) for this range  $q > \tau + 2$ .

For a regular lattice graph  $G$  it is of interest to investigate the dependence of  $W(\{G\}, q)$  on the vertex degree (coordination number)  $d$ . This study was carried out in [8] [37, 38], and it was shown that  $W(\{G\}, q)$  is a non-increasing function of  $d$ . This is understood as being a consequence of the fact that (except for tree graphs, which are not relevant here), roughly speaking, increasing the vertex degree tends to increase the constraints on a proper  $q$ -coloring of the vertices and therefore tends to decrease  $W(\{G\}, q)$ . One is also motivated to investigate the same question with the families of planar triangulations under study here. However, since  $d_{eff} = 6$  for a family of planar triangulation graphs one is limited to a fixed  $d_{eff} = 6$  and hence cannot carry out the type of comparative study involving a variation in  $d_{eff}$  that was performed in [8], [37, 38]. In [1] and the present work, we have found that families of planar triangulations can have different  $W(\{G\}, q)$  functions. This is consistent with the results in [8]-[38]. Indeed, one has already encountered examples of this. For example, the square and kagomé ( $3 \cdot 6 \cdot 3 \cdot 6$ ) lattices both have the same vertex degree, namely 4, but they have different  $W$  functions, and similarly, the honeycomb, ( $3 \cdot 12^2$ ), and ( $4 \cdot 8^2$ ) lattices have the same vertex degree, namely 3, but again, they have different  $W$  functions [8, 37, 38].

## XX. COMPARATIVE DISCUSSION

In this section we give a comparative discussion of some limiting quantities for the various families of planar triangulations that we have studied so far. For one-parameter families of planar triangulations  $G_{pt,m}$  for which  $P(G_{pt,m}, q)$  is of the form (2.8) we have proved that  $r(G_{pt,\infty}) = 0$  and have investigated the various values of  $a_{G_{pt}}$  defined in (3.18). This constant is strictly less than unity, and it is of interest to see which families yield larger and smaller values of  $a_{G_{pt}}$ . We display the values that we have obtained in Table VII. As is evident, in the set of  $j_{max} = 1$  families of planar triangulations, the family of cylindrical strips of the triangular lattice,  $TC_m$  (equivalently, iterated octahedra) yields the largest value of  $a_{G_{pt}}$ , which is within 9 % of its upper bound of 1. In the  $j_{max} = 3$  families, the one that yields the largest value of the limiting ratio  $r(G_{pt,\infty})$  is the family,  $B_m$ , with  $r(B_\infty) = 0.6180..$ . A second type of asymptotic limiting function is  $W(\{G\}, q)$ . We have given a comparative analysis of this in the previous section.

We have also investigated the values of  $P(G_{pt,m}, q)$  at  $q = \chi(G_{pt,m})$  for the families of planar triangulations that we have studied. Recall the definition that a graph  $G$  is  $k$ -critical iff  $\chi(G) = k$  and  $P(G, k) = k!$ . We find a variety of behavior. For example, (i)  $\chi(R_m) = 4$  and  $P(R_m, 4) = 4!$ , so  $R_m$  is 4-critical; (ii)  $\chi(TC_m) = 3$  and  $P(TC_m, 3) = 3!$ , so  $TC_m$  is 3-critical; but (iii)  $\chi = 4$  for  $I_m$ ,  $H_m$ ,  $L_m$ ,  $D_{m-4,3}$ , and  $F_{m \geq 3}$  but none of these families is 4-critical. For other families  $G_{pt,m}$ , the chromatic number depends on whether  $m$  is even or odd. For example, for even  $m$ ,  $\chi = 3$  for  $B_m$  and  $D_{m-4,2}$ , and these graphs are 3-critical, while for odd  $m$ ,  $\chi = 4$  for  $B_m$  and  $D_{m-4,2}$ , but neither of these graphs is 4-critical.

There are a number of further directions in which the present research could be extended. One could, for example, study Tutte polynomials, or the equivalent, Potts model partition functions  $Z(G_{pt}, q, v)$ , for planar triangulation graphs  $G_{pt}$  (where  $v$  is a temperature-like Boltzmann variable). However, a number of the special properties that make the chromatic polynomials of these graphs amenable to analysis do not generalize to the full Potts model partition function. These include, for example, the use of the complete-graph intersection theorem and the property that the chromatic polynomial has  $q(q-1)(q-2)$  as a factor. For example, for the triangle graph  $K_3$  itself, the Potts model partition function  $Z(K_3, q, v) = (q+v)^3 + (q-1)v^3 = q(q^2 + 3qv + 3v^2 + v^3)$  only has the factor  $q$ . Another direction of investigation would be to calculate weighted-set chromatic polynomials [39, 40] and Potts model partition functions with an external magnetic field that favors or disfavors a single value of  $q$  or a set of  $s$  such values [41] for planar triangulation graphs. Work on this is underway.

TABLE VII: Some asymptotic limiting quantities for one-parameter families of planar triangulations. The shorthand notation  $3me, 4mo$  means  $\chi = 3$  if  $m$  is even and  $\chi = 4$  if  $m$  is odd. Additional information for  $\chi$  values is  $\chi(R_3) = 3$ ,  $\chi(F_1) = 4$ , and  $\chi(F_2) = 3$ . Numerical values are quoted to three significant figures.

$G_{pt,m}$	$n(G_{pt,m})$	$\chi(G_{pt,m})$	$j_{max}$	$r(G_{pt,\infty})$	$a_{G_{pt}}$
$R_m$	$m + 2$	4 if $m \geq 2$	1	0	$(-1 + \sqrt{5})/2 = 0.618$
$TC_m$	$3m$	3	1	0	$(3 - \sqrt{5})^{1/3} = 0.914$
$I_m$	$9m + 3$	4	1	0	$[(-315 + 141\sqrt{5})/2]^{1/9} = 0.8055$
$G_{CM,m}$	$8m + 3$	3	1	0	$[(115 - 51\sqrt{5})/2]^{1/8} = 0.885$
$F_m$	$m + 4$	4 if $m \geq 3$	3	0	0.786
$B_m$	$m + 2$	$3me, 4mo$	3	$(-1 + \sqrt{5})/2 = 0.618$	1
$H_m$	$m + 5$	4	3	$(7 - 3\sqrt{5})/2 = 0.146$	1
$L_m$	$m + 5$	4	3	$-2 + \sqrt{5} = 0.236$	1
$D_{m-4,0}$	$m + 5$	$3me, 4mo$	3	$-4 + 2\sqrt{5} = 0.472$	1
$D_{m-4,1}$	$m + 6$	4	3	$(-15 + 7\sqrt{5})/2 = 0.326$	1
$D_{0,m-2}$	$m + 7$	$3me, 4mo$	3	$-13 + 6\sqrt{5} = 0.416$	1
$D_{1,m-2}$	$m + 8$	4	3	$-22 + 10\sqrt{5} = 0.361$	1
$D_{m,m}$	$2m + 9$	$3me, 4mo$	6	$(3 - \sqrt{5})/2 = 0.382$	1
$S_{m,m}$	$m + 7$	4	4	0	$(-1 + \sqrt{5})/2 = 0.618$

## XXI. CONCLUSIONS

In this paper we have presented an analysis of the structure and properties of chromatic polynomials  $P(G_{pt,\vec{m}}, q)$  of families of planar triangulation graphs  $G_{pt,\vec{m}}$ , where  $\vec{m} = (m_1, \dots, m_p)$  is a vector of integer parameters. We have discussed a number of specific families with  $p = 1$  and  $p = 2$ . These planar triangulation graphs form a particularly attractive class of graphs for the analysis of chromatic polynomials because of their special properties. One of these is the fact that when evaluated at  $q = \tau + 1$ , the chromatic polynomial of a planar triangulation graph satisfies the Tutte upper bound (1.1). We have studied the ratio of  $|P(G_{pt,\vec{m}}, \tau + 1)|$  to the Tutte upper bound  $(\tau - 1)^{n-5}$  and have calculated limiting

values of this ratio as  $n \rightarrow \infty$  for various families of planar triangulations. We also have used our calculations to study zeros of these chromatic polynomials. Among our results, we have shown that if  $G_{pt,\vec{m}}$  is a planar triangulation graph with a chromatic polynomial  $P(G_{pt,\vec{m}}, q)$  of the form (8.1), then (i) the coefficients  $c_{G_{pt},\vec{i}}$  must satisfy a number of properties, which we have derived; and (ii)  $P(G_{pt,\vec{m}}, q)$  has a real chromatic zero that approaches  $(1/2)(3 + \sqrt{5})$  as one or more  $m_i \rightarrow \infty$ . We have constructed a  $p = 1$  family of planar triangulations with real zeros that approach 3 from below as  $m \rightarrow \infty$ . A one-parameter family  $F_m$  with  $j_{max} = 3$  and nonpolynomial  $\lambda_{F,j}$  has been studied. We have also presented results for a number of results for chromatic polynomials of various two-parameter families of planar triangulations. Implications for the ground-state entropy of the Potts antiferromagnet are discussed. Our results are of interest both from the point of view of mathematical graph theory and statistical physics and further show the fruitful connections between these fields.

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